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GEOMETRY

PRACTICAL AND THEORETICAL

VOL. III. SOLID GEOMETRY

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PRACTICAL AND THEORETICAL

PARI PASSU

IN THREE VOLUMES

BY

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VOL. III. SOLID GEOMETRY

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PREFACE

"If you want to learn a subject, teach it," and "If you want to learn a subject, write a book on it." Both are very sound pieces of advice; but, as far as Elementary Solid Geometry is concerned, they have not been hitherto very easy to carry out. The would-be learner should have access to the great fund of information that exists as the outcome of centuries of work and of the ideas of many a genius; he must be able to apply what he has learnt to everyday problems (always increasing in number with the advance of knowledge); he must have experience as well and plenty of time at his disposal. Access of a kind there has doubtless been for a long while, but the stores of information are either contained in many books, none of them fully intelligible alone, or else concentrated, as in an Encyclopaedia in a bulky and costly form: and, though countless examples may be culled from Nature and from everyday affairs, yet it is extremely easy to pass them by unseen. There is therefore still something wanting, which it is hoped that this volume will supply.

My object is to present elementary features of Solid Geometry, with many ideas as to their application, in the hope that the practice will help to find other uses for the truths, which have been known so long and yet are always fresh.

The subject is developed both from the practical and theoretical point of view. Reference is made continually to the historical side. In the specimen examples the tendency has been to note how the ground should be prepared for the solution of the problem, by this means emphasizing the necessary lines of attack. There

are 10 propositions and about 450 examples (the scope of which is indicated by the following rough summary):

5% Plans and Elevations (G.D.),
35% Special cases (G.D. or Calculations),
38% General cases (Theoretical Calculations),
22% Riders.

They have been very clearly graded. There are no hard and fast lines as to the beginning and end of the subject, so certain sections of it, about the inclusion of which opinions may differ, are given in Appendices I. and II.; in the former is a note on some ways of showing relief, in the latter are five propositions somewhat on the border line. The determination of the particular style of the diagrams, which are a feature of the book, has been difficult in many cases. Unquestionably models convey correct ideas of 3 dimensions much more quickly to the eye than pictures on flat paper (necessarily in 2 dimensions); but their use is impracticable in a text-book. It is possible to adopt various plans on stereoscopic principles, with special instruments for viewing every figure, but these are more likely to excite wonder by their realism than to afford any genuine aid in the acquisition of the habit (so essential to the student of Solid Geometry) of imagining, and also depicting, space by flat diagrams alone. Here shading and colour help; however their reproduction requires some artistic skill and is quite prohibitive in the matter of cost: this alternative is seldom used. In certain cases the camera has been of service, and reproductions of photographs are given on several occasions. The adoption of lines of different thicknesses conveys some idea of solidarity, while conventional lines of shading are frequently appropriate. In some figures the *important* lines have been emphasized.

My thanks are due to the Controller of His Majesty's Stationery Office for permission to include some questions from recent Army Entrance Examination Papers. The advice of Mr. A. R. Hinks, Mr. A. L. Onslow and Mr. E. V. Slater, on various points, has

been of great benefit. I am indebted to Mr. D. F. Landale for a paper model of a regular icosahedron, from which the photograph on page 165 was made. The criticisms of Mr. F. W. Dobbs and Mr. W. D. Eggar have been most valuable. Many others have consciously, or unconsciously, given me information and ideas; to them I am very grateful, and amongst these I should mention (especially) important officials in the Cunard Steamship Company and the L. & N.W. Railway Company. Again, in the production of this volume, as well as of its predecessors, I owe an enormous debt of gratitude to Mr. W. Hope-Jones, who throughout has placed his knowledge at my disposal, has frequently bestowed considerable labour on a diversity of affairs, has criticised in very great detail, and has ended by checking every answer. To the careful and thorough revision, and suggestions, of the General Editor I should like to pay a warm tribute; and last, but not least, I am much beholden to the Publishers, who have had far from an easy task.

V. L. N. F.

ETON, 1922.

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SUMMARY OF PROPOSITIONS

Briefly the Propositions are :

NOTES

Much of this advice does not refer to Solid Geometry alone. It is suggested that you read through the Notes from time to time; it is nearly useless reading them only once preceding all study.

§ **1.** In numerical work, generally speaking, keep a greater degree of accuracy in the computations than is required in the result. It is often dangerous to make too drastic approximations early.

§ **2.** On principle avoid mere computation as long as possible. For instance it is quicker, and more accurate, to use $\sqrt{2}$ than 1·41421... and such like; it is far easier for you to check your work if you have avoided numerical computation; by all means shorten your work, *en route*, by obvious simplifications, but any modification at all elaborate is generally better postponed. Above all it is far clearer to those you are trying to convince with your arguments.

The notation required nowadays is vastly simplified. The convenient processes of Arithmetic, Algebra, Trigonometry and the Calculus are no longer in the water-tight compartments into which they were separated in the past.

It should be noted that it is questionable to give a result to a greater degree of accuracy than the data perhaps warrant.

§ **3.** Do not try to run before you can walk; so be content, in the first reading, to do the work very thoroughly (but thorough-

ness is not synonymous with long-windedness). Quickness is clearly much to be sought for, ***but not at first***; you should learn to be quick when you have grasped the principles fully, and that is certainly not when you have only a hazy notion about the subject.

§ **4.** Be inclined to use makeshift models to help to give you ideas for the solution of a problem in three dimensions. Frequently stiff paper bent, a set-square leaning against a book, a leg of a pair of dividers stuck into a board (and such-like devices), will illustrate the problem and tell you readily which sections to draw, and that is very much more than half the battle.

§ **5.** In any particular case you should adopt the style of diagram which will best explain the point you wish to make; naturally with due reference to your powers as a draughtsman and to the time at your disposal.

§ **6.** One Science is the handmaid of another; and it is with co-operation (each doing his own share) that you can make light of the difficulties which seem to be so impossible to a single individual.

§ **7.** It has been said that $\frac{9}{10}$ of every business, or profession, is common sense. Solid Geometry affords abundant opportunities for the exercise of that. The remaining $\frac{1}{10}$ (technical details) will be easy enough, when the time comes, if your grasp of the $\frac{9}{10}$ is sound.

§ **8.** The application of Solid Geometry to the Earth is of very great practical importance to human beings.

TO THE STUDENT.

Throughout, the bookwork and examples are either blank, o have one asterisk * at the side, or two asterisks * * at the side.

It is suggested that in the first reading you should confine your ideas to the unmarked parts. On a second reading you should also tackle the * parts. On a third reading you should be familiar with all.

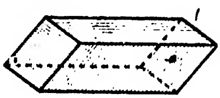
The examples preceded by the letter O might be discussed orally. Answers are not given to those questions.

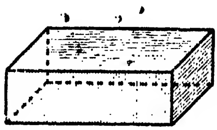
In the "special cases" calculations should give quicker (and more reliable) answers than those obtained by geometrical drawing alone: but clearly using one method as a check to the other is safer still.

CHAPTER XL.

PARALLELEPIPEDS AND TETRAHEDRA.

§ 1. The shape of a **Parallelepiped** is most easily realized from a model, in default of which use the two figures following. A parallelepiped has a parallelogram as base. Its four side faces are parallelograms. Its opposite faces are parallel and the same. It has six faces. It may be oblique or right, but in the latter (special, but very important) case it is sometimes called a *cuboid*; [a box and a room are familiar examples].

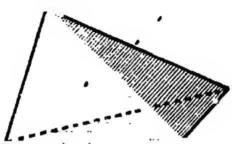




The most important right-parallelepiped is the **cube**, which has all its edges equal and its faces are squares.

The erroneous spelling parallelopiped is often found. The derivation of the word includes the Greek preposition ἐπί (epi), so that the "o" spelling is incorrect.

§ 2. A **Tetrahedron** is the solid bounded by 4 plane faces (the minimum number of faces for a solid). Each face is a triangle. Two important properties of these solids are embodied in Propositions **39** and **40**.



A tetrahedron is sometimes called a **Pyramid with a triangular base.**

§ **3.** It should be noted that **a plane is determined by any one of the following :**

(1) *Three points not in the same straight line* (hence a triangle determines a plane).
(2) *Two intersecting straight lines.*
(3) *Two parallel straight lines.*
(4) *A straight line and a point not in it.*

Thus, if any one of the above conditions obtains in any particular case, we can apply two-dimensional geometry for that portion.

These are familiar facts of everyday life. For (1) you will recollect that it is *always* possible to rest a 3-legged table on the floor; but that, if the table has more than 3 legs, they must be of correct lengths.

For (2) and (4) you should imagine a plane, through a line in question, revolved about until it satisfies the other condition.

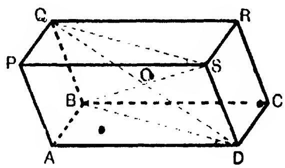
For (3) you might remember that a tea-tray, if it slides downstairs, can be in contact with the parallel front edges of two steps of a *straight* flight, but not where the flight goes round a corner, for there the front edges are no longer parallel.

§ **4.** In Propositions **38** and **39** it is assumed that **straight lines parallel to the same straight line (not in the same plane with them) must be parallel to one another.** If the truth of this important fact is not taken as a piece of common knowledge, reference should be made to Proposition W in Appendix II. on page 580.

PROPOSITION 38.

§ 5. **General Enunciation.** *The (four) diagonals of a parallelepiped are concurrent and bisect each other.*

Particular Enunciation. PQRSABCD is the parallelepiped. The diagonals are PC, QD, RA and SB (only 2, QD and SB, are shown in the figure). To prove that PC, QD, RA and SB are concurrent and bisect each other.



Proof. QB and SD are equal and parallel (for each is equal and parallel to PA).*

∴ QSDB is a parallelogram,

so that its diagonals (QD and SB) meet and bisect each other. Similarly each of the 3rd and 4th diagonals of the parallelepiped meets and bisects the 1st.

So that the 4 diagonals PC, QD, RA and SB are concurrent and bisect each other.

Q.E.D.

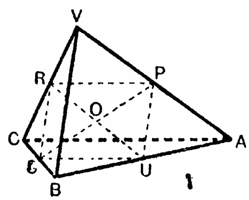
[The intersection of the diagonals is often marked O.]

* See § 4 of this chapter.

PROPOSITION 39.

§ 6. **General Enunciation.** *The joins of the mid points of the opposite edges of a tetrahedron are concurrent and bisect each other.*

Particular Enunciation. VABC is the tetrahedron. P, Q, R, S, T and U are the mid points of the 6 edges VA, VB, VC, BC, CA and AB (only 4, namely P, R, S and U, are shown in the figure). To prove that PS, QT and RU are concurrent and bisect each other.



Proof. In the $\triangle$AVB, P bisects AV and U bisects AB;

$\therefore$ PU $= \frac{1}{2}$VB and is parallel to it.

Similarly, in the $\triangle$CVB, RS $= \frac{1}{2}$VB and is parallel to it.

$\therefore$ PU and RS are equal and parallel; *

$\therefore$ PUSR is a parallelogram,

so that its diagonals PS and RU meet and bisect each other; similarly QT and RU meet and bisect each other;

$\therefore$ PS, QT and RU are concurrent and bisect each other.

Q.E.D.

[The intersection is often marked O.]

* See § 4 of this chapter.

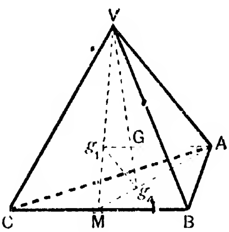
§ 7. It is interesting to observe that the six points (P, Q, R, S, T and U), in the preceding, form the angular points of an octahedron. (It is usual only to consider octahedra, whose three diagonals are concurrent and bisect each other.) It is easy to imagine such an octahedron nestling in the midst of *any* tetrahedron; reference should be made to the figure of Question **24** on page 475, though this shows one big *regular* tetrahedron split up into four little tetrahedra with one little octahedron in their midst, yet it is quite possible to generalise. You should also note Chapter XLIII., the end of § **5** (on page 471) for another important property, generalising again.

PROPOSITION 40.

§ 8. **General Enunciation.** *The straight lines which join the (four) vertices of any tetrahedron to the centres of gravity (or centroids) of the opposite faces are concurrent, and divide each other in the ratio* 3 : 1.

Particular Enunciation. VABC is the tetrahedron. g_1, g_2, g_3 and g_4 are the centres of gravity of the faces VBC, VCA, VAB and ABC (only g_1 and g_4 are shown in the figure).

To prove that Ag_1, Bg_2, Cg_3 and Vg_4 are concurrent, and divide each other in the ratio 3 : 1.



Construction and Proof. M is the mid point of the edge BC; hence, in $\triangle$ VBC (since g_1 is its centre of gravity), Mg_1 = $\frac{1}{3}$MV; and, in $\triangle$ ABC (since g_4 is its centre of gravity), Mg_4 = $\frac{1}{3}$MA; so that, in $\triangle$ MVA, Mg_1 : MV = 1 : 3 = Mg_4 : MA;

$\therefore$ g_1g_4 is parallel to VA and equal to $\frac{1}{3}$VA.

Now MAV is a plane, so that Ag_1 and Vg_4 cut.

Let them cut at G.

In the similar triangles VGA and g_4Gg_1,

VG : g_4G = VA : g_4g_1 = 3 : 1;

hence Ag_1 and Vg_4 meet and divide each other in the ratio 3 : 1.

In the same way the second and third, Bg_2 and Cg_3, meet and divide the first, Ag_1 (and are divided by it), in the same ratio.

$\therefore$ Ag_1, Bg_2, Cg_3 and Vg_4 are concurrent and divide each other in the ratio 3 : 1. **Q.E.D.**

§ 9. The truth of the preceding proposition can be inferred from statical considerations.

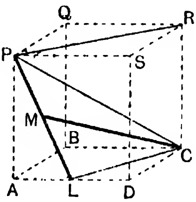
For, with the same figure, the C.G. of the tetrahedron VABC is the same as that of equal particles situated at its 4 vertices.

Suppose that the particles are each of unit mass. The three particles, at A, B and C, have the same effect as a single particle, of mass 3, at g_4 (the C.G. of $\triangle$ABC).

Again, the two particles (1 at V and 3 at g_4) have the same effect as a single particle 4 at G (where VG : G$g_4 = 3 : 1$); so that G is the C.G. of the tetrahedron.

Now the tetrahedron has only *one* C.G.; and, by treating different vertices of the tetrahedron as the single vertex, we must get the same C.G., and this proves the proposition.

§ **10.** The propositions, in this chapter, have resolved themselves into taking appropriate *sections* and working in *two dimensions*. As an **Example** of calculation consider the following, observing the figure for the meaning of the letters.



L is the mid point of DA. (The edges of the cube are a.) M is the mid point of the line PL. How long is CM ?

[*One way* is to work with the triangle PCL. CM is a median of that triangle. So that the first thing to do is to calculate the squares on the sides of that triangle.]

Join PR, PC and CL.

Plane **PQRS** $\triangle$PQR, $PR^2 = PQ^2 + QR^2$ (Pyth.);
$\therefore\ PR^2 = 2a^2$.

Plane **PRCA** $\triangle$PRC, $PC^2 = PR^2 + RC^2$ (Pyth.);
$\therefore\ PC^2 = 3a^2$.

Plane **ABCD** $\triangle$CDL, $CL^2 = CD^2 + DL^2$ (Pyth.)
$= a^2 + \frac{a^2}{4} = \frac{5}{4}a^2$.

Plane **PSDA** $\triangle$PAL, $PL^2 = \frac{5}{4}a^2$ (similarly);
$\therefore\ PM^2 = ML^2 = \frac{5}{16}a^2$.

Plane **PCL** CM is a median of $\triangle$CPL ;
$\therefore\ 2CM^2 + 2PM^2 = PC^2 + CL^2$;
$\therefore\ 2CM^2 + \frac{5}{8}a^2 = 3a^2 + \frac{5}{4}a^2$;
whence $CM^2 = \frac{29}{16}a^2$;
$\therefore\ CM = \frac{1}{4}a\sqrt{29}$.

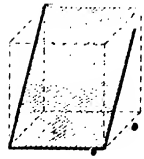
[The Student should understand *another way*, reckoning CM as a diagonal of a right-parallelepiped, with 3 edges

CD(a), along DA($\frac{3}{4}a$) and through M parallel to AP($\frac{1}{2}a$):
thus $CM^2 = a^2\{(1)^2 + (\frac{3}{4})^2 + (\frac{1}{2})^2\}$ etc.]

EXAMPLES 40 a. (CALCULATIONS. GENERAL CASES.)

1. A swimming bath is a long and b wide, its depth varies uniformly from s at the shallow end to d at the deep end. What is its volume ?

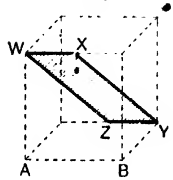
2. A cube (of edge a) is cut in half, diagonally, by a plane through two opposite edges. What is the shape of this section? What is the area of this section?



3. X and Z are the mid points of two opposite edges of a cube (of edge a), and W and Y are diagonally opposite corners of the cube adjacent to X and Z.

Prove that WXYZ is a parallelogram. What is its area?

[A and B are needed for Question **16**.]

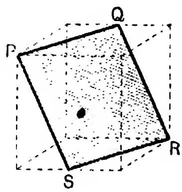


4. OA, OB and OC are three straight lines mutually at right angles. Prove that the sum of the squares on the sides of the triangle ABC is equal to twice the sum of the squares on OA, OB and OC.

5. A rectangular parallelepiped has edges a, b, c. What is the length of its diagonals?

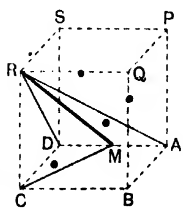
What is the diagonal of a cube whose edge is a? [You should commit this answer to memory.]

6. The quadrilateral PQRS is formed by joining two diagonally opposite corners of a cube and the middle points of two opposite edges, as shown in the figure. Show that PQRS is plane and that it is a rhombus in shape. If the edges of the cube are a, what are the diagonals of PQRS, and what is its area?



7. M is the middle point of the edge AD of a cube. R is the corner of the cube diagonally opposite to A. Calculate the length of RM in three ways. The edges of the cube are a.

[Hints. (1) Calculate RM in the right-angled triangle RDM. (2) Calculate MC in the right-angled triangle CDM, and then RM in the right-angled triangle RCM. (3) Consider RM as a median of the triangle RDA.]



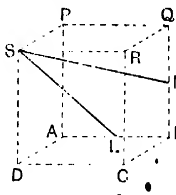
8. What is the area of the triangle formed by joining the mid point of one edge of a cube to the ends of the opposite edge? The edge of the cube is a.

9. What are the lengths of the straight lines joining a corner of a cube to the points of trisection of one of the three opposite edges of the cube? The length of an edge of the cube is a.

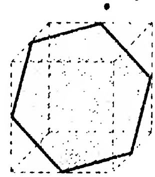
10. A room is a long, b wide and c high. What is the area of the triangle formed by joining one-corner-of-the-ceiling to ends of the length-of-the-floor on the opposite side of the room? Write down the area of triangle if the joins are to the width-of-the-floor, again furthest away.

11. A pyramid has an equilateral triangle (with sides a) as base. Its other 3 edges are each of length l. Calculate the height of the pyramid.

12. Find the angles between diagonals of a cube.



* **13.** If L and M the mid points of two edges BA and BQ (of a cube), are joined to the point S (S being the corner diagonally opposite B), find the size of the angle LSM.



* **14.** The figure suggests how a cube may be cut to give a regular hexagon. Give the construction and reasons fully (but no formal proof is required). If an edge of the cube is a, what is the side of the hexagon? What is the volume of each of the portions into which the cube is cut? What is the area of the hexagon?

* **15.** If all the edges of a tetrahedron are equal to a, how far is it between its centre of gravity and one of its vertices?

* **16.** In the figure of Question 3, if K is a point in XY such that $XK = \frac{1}{4}XY$, give the following lengths WK, AK and BK.

* **17.** VABC is a tetrahedron in which $VA = l$, $VB = m$, $VC = n$, $BC = a$, $CA = b$ and $AB = c$. What is the length of the line joining the middle points of VA and BC? Write down (by cyclic changes) the lengths of the other two lines joining the middle points of opposite edges.

[Hint. Work with a median of various triangles.]

*18. A room measures a by b by c. Find the angles between the diagonals of the room. [3 cases.]

[*N.B.*—By a diagonal of a room is meant a line joining one corner of the ceiling to the corner of the floor furthest from it.]

*19. A pyramid with a square base has four equal isosceles triangular sides. The height of the pyramid is equal to $\frac{1}{2}$ the side of the square. What are the angles of one of the isosceles triangles?

20. With the same notation as in Question **17, if g_1 is the centre of gravity of the face ABC, how long is Vg_1, and how long is VG? G is the centre of gravity of the solid.

[HINT. See the figure of Prop. **40** on page 428. Calculate VM (a median of $\triangle$VBC) and AM (a median of $\triangle$ABC). Then work with $\triangle$VMA.]

**21. A door, a high and b wide, is ajar at an angle 2θ. At what angle is a diagonal of the open door inclined to the *same* diagonal of the door when it is shut?

**22. Referring to the figure on page 431, AR is a diagonal of a cube whose edges are a. AR is trisected at the points H and K. What are the lengths of DH and DK?

EXAMPLES 40 b (RIDERS).

1. Prove that the area of the triangle formed by joining the mid points of three edges of a tetrahedron (which meet at one vertex) is $\frac{1}{4}$ the area of the face, which is opposite to the vertex at which those edges meet.

2. Equal lengths, QX and DZ, are taken (internally) along one diagonal QD of a parallelepiped. Also equal lengths, BW and SY, are taken (internally) along another diagonal BS. Prove that WXYZ is a parallelogram.

[HINT. Make a figure like that of Proposition **38** on page 425.]

3. ABCD is a rectangle whose diagonals cut at K. P is any point in space. Prove that $PA^2 + PB^2 + PC^2 + PD^2 = 4PK^2 + AC^2$.

4. Show that the square on a diagonal of a cube is 3 times the square on any of its edges.

5. Prove that the mid points of the sides of a skew (*i.e.* not plane) quadrilateral must be the vertices of a parallelogram.

*** 6.** Show that the sum of the squares on the 4 diagonals of a parallelepiped is equal to the sum of the squares on its 12 edges.

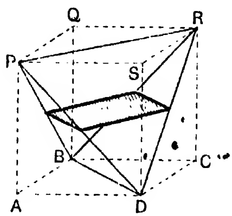
[HINT. Recollect that the diagonals must bisect each other, and work with medians of triangles.]

*** 7.** If a parallelepiped is cut by a plane which intersects two pairs of opposite faces, the common section forms a parallelogram.

*** 8.** Three points A, B, C are taken one on each of three conterminous edges of a cube: prove that the angles of the triangle ABC are all acute.

[HINT. If the conterminous edges of the cube meet at O, let OA, OB and OC be respectively x, y and z. Calculate BC^2, CA^2 and AB^2, and prove $BC^2 < CA^2 + AB^2$; similarly, etc.]

*** 9.** The dotted figure PQRSABCD is a cube. Its corners P, R, B and D are joined to form a triangular pyramid. Prove that all its edges must be equal. Four (of its six) edges are bisected (two opposite edges are excluded). Prove that a square must be formed. (Naturally the shaded figure looks like a parallelogram only.) If the edges of the cube are a, what is the area of the square?



*** 10.** In the figure of Proposition **38** on page 425, if PA is bisected at K, and RC at L, prove that KQLD must be a parallelogram.

*** 11.** The sum of the squares on the edges of any tetrahedron is four times the sum of the squares on the joins of the mid points of opposite edges.

*** 12.** Three straight lines AOB, POQ and XOY meet and bisect each other at O. Show that the sum of the squares on the 12 edges of the octahedron (of which they are diagonals) is twice the sum of the squares on those diagonals.

*** 13.** Prove that if a tetrahedron is cut by a plane parallel to two opposite edges, the section must be a parallelogram.

*** 14.** VABC is any tetrahedron. The edges VA, VB and VC are bisected at X, Y and Z respectively. Prove that the radius

of the circumcircle of the triangle XYZ is equal to the radius of the circle which passes through the middle points of the sides of the triangle ABC.

* **15.** Give a geometrical construction for drawing a straight line equally inclined to three straight lines which meet in a point, but are not in the same plane.

* * **16.** Referring to the figure on page 430, prove that the two planes PDB and CQS must trisect the diagonal AR.

* * **17.** V is a vertex of any tetrahedron VABC. K is the centre of gravity of the face ABC. Prove that

$$VA^2 + VB^2 + VC^2 = 3VK^2 + AK^2 + BK^2 + CK^2.$$

[HINT. If AK produced cuts BC in L, recollect that VL is a median of the triangle VBC.]

* * **18.** ABC is a triangle. Q is any point in space. G is the centre of gravity of the triangle. Prove that

$$AQ^2 + BQ^2 + CQ^2 = AG^2 + BG^2 + CG^2 + 3GQ^2.$$

[You should note the mathematical identity of Questions **17** and **18**; and (after drawing the figure) should omit **18** if you have had no difficulty in **17**.]

* * **19.** ABCD is any tetrahedron and G is its centre of gravity. Prove that

$$BC^2 + CD^2 + DB^2 + AB^2 + AC^2 + AD^2 = 4\{AG^2 + BG^2 + CG^2 + DG^2\}.$$

* * **20.** OX, OY and OZ are three straight lines mutually at right angles. They are taken as axes from which to define the position of any point. OP is a line in space, such that $\angle XOP = \alpha$, $\angle YOP = \beta$ and $\angle ZOP = \gamma$. Prove that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

[*N.B.*—These cosines are called the "Direction-Cosines" of the line OP, a most vital idea in Analytical Solid Geometry. For the solution of the problem you might consider OP as a diagonal of a rectangular parallelepiped whose edges are along OX, OY and OZ.]

CHAPTER XLI.

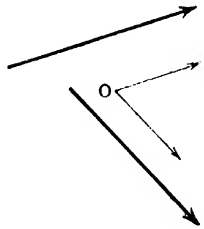
LINES AND PLANES.

At first the essential things in this chapter are (1) *the* **Inclination of 2 non-intersecting lines (not in the same plane),** (2) *the* **Inclination of two planes** *and* (3) *the* **Inclination of a line to a plane.** *It is suggested that you should consider some exercises after reading* §§ **1** *to* **5.**

§ **1.** In Plane Geometry we have already considered the angle between two intersecting straight lines (for they are necessarily in the same plane); and we have also discussed the case of parallel straight lines (again necessarily in the same plane). There remains the case of the **Inclination of two straight lines which do not meet, and which yet are not parallel** (not being in the same plane); such as the parapet of a bridge crossing over (and above) railway lines. We are quite familiar with the idea of the bridge being at right angles, for instance, to the track of the railway lines, understanding that other straight lines (parallel to them and actually in the same plane) meet at right angles. As another instance, if two aeroplanes are travelling, one climbing *up* towards the North and the other gliding *down* towards the North-East, the angle between their courses may be considered even though their paths do not cut.

For the angle between two lines, which cannot meet, the essential thing is to get their directions by lines (parallel to them) which

actually do meet. It is the angle between their directions that is required.

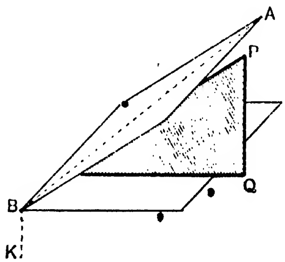


Thus, for this angle between two lines (whether they cut or not), it is sufficient to take any convenient point O, and to draw (through O) lines parallel to each of the given lines, and to consider the angle between these parallels.

The inclination of two lines, which *are not* in the same plane, is simply the inclination of two lines, parallel to them, which *are* in the same plane.

As a further illustration:

A set-square is introduced between the leaves of a book, opening it. If the inclination of PQ (a side of the set-square) to AB (a



diagonal of the cover of the book) is to be considered [in a practical case, it is not a bad plan to chalk AB], it is only necessary to imagine BK, parallel to PQ, and to consider the angle between BK and AB.

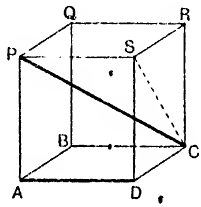
§ 2. As an **Example** of calculation consider the following:

For a cube, find the angle between one of its diagonals and an edge which does not meet that diagonal.

PC is the diagonal and AD the edge.

[Now, since the *direction* of PS is the same as that of AD, all that is necessary is to consider the angle between PC and PS.]
[So work with the triangle PSC.]

Join SC.



If the edges of the cube are a,

PS $= a$ (an edge of the cube);

SC $= a\sqrt{2}$ (a diagonal of a face of the cube);

and $\angle$PSC is a right angle (for PS is perpendicular to each of the *two intersecting lines* SR and SD, at their point of intersection S, and is therefore perpendicular to the line SC, which passes through their intersection S and which is in the plane that they determine).*

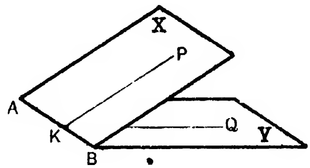
Hence tan SPC $= \sqrt{2}$ or about 1·4142, and then trigonometrical tables give the angle as about 54° 44′.

(Drawing to scale should give good results too.)

* If the truth of this is not assumed reference should be made to Proposition V in Appendix II. on page 578.

§ 3. The **Inclination of two Planes** to each other is measured as follows:

X and **Y** are the planes. AB is their line of intersection. K is *any* point on AB. Draw KP in one plane and KQ in the other,



each perpendicular to AB. The angle between the *lines* KP and KQ is the angle between the *planes* **X** and **Y**.

[It should be noted that the same angle is obtained whatever point K, on AB, is chosen.]

Also the angle between two planes is often measured by the angle between their normals. This is most important.

N.B.—A **normal** to a plane is a straight line perpendicular to the plane. (Knowing the direction of a normal to a plane, we know, in a certain way, the direction of the plane itself.)

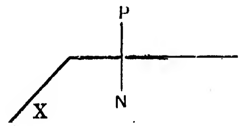
By far the commonest case (in practice) is the consideration of the inclination of a plane to the *horizontal*. Gravity has such an overwhelming effect on us that "upright" and "horizontal" often go without saying; for instance, it is often understood that walls are *upright*, and the surface of a mill-pond is proverbially *horizontal*, and by the slope of a roof is certainly meant the angle between it and the *horizontal plane*.

In the previous figure, if the plane **Y** is horizontal, PK is called the **Line of greatest slope** in the plane **X**. (You should note Question **11** on page 457 in this connection.)

Two planes meeting are said to form a **dihedral angle**.

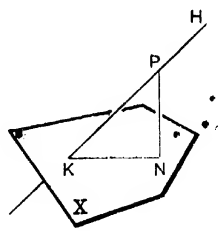
§ **4.** Many extensions of ideas gathered in Plane Geometry follow.

The **Projection of a point on a plane** is the foot of the perpendicular from the point to the plane. In the figure, N is the projection of P on the plane **X**.



Planes are parallel if they can never meet, even when produced.
A **line is parallel to a plane** under the same circumstances.
Points in the same plane are said to be **coplanar.**

§ **5.** The **Inclination of a line to a plane** is measured in this way: HK is the line meeting the plane **X** at the point K. P is *any* point on HK. PN is perpendicular to **X**. KN is joined. The angle between the line HK and the plane **X** is PKN. (You should note Question **12** on page 457 in this connection.)



It should be observed that the angle between the line and the plane is the complement of the angle between the line and a normal to the plane.

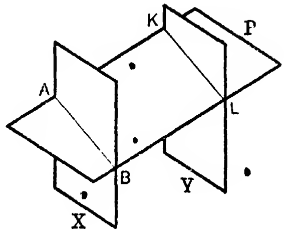
[The problem of constructing a perpendicular to a plane, from a given external point, is discussed in Proposition **43** on page 443.]

N.B.—When a ray of light falls on a surface, the angle of incidence is equal to the angle of reflection. It is unlikely that we should consider the angle between the ray of light and a wrong line in the plane, but any ambiguity is avoided if the angles of incidence and of reflection are measured from the normal (as is correct). The plane containing these two rays is necessarily perpendicular to the plane of the surface.

PROPOSITION 41.

§ **6. General Enunciation.** *If a plane cuts two parallel planes it must cut them in parallel lines.*

Particular Enunciation. The plane **P** cuts the parallel planes **X** and **Y** in the straight lines AB and KL. To prove AB and KL are parallel.



Proof. The planes **X** and **Y** are parallel (given).

∴ lines in them, such as AB and KL, cannot meet.

Now AB and KL *cannot meet and are in the one single plane* **P**;

∴ AB and KL are parallel.

Q.E.D.

§ **7.** The two following facts are obvious, and no proof is given of them in this book. Propositions, substantiating their truth, are proved explicitly by Euclid (Book XI. Propositions 6 and 8), of course, on his assumptions.

If two straight lines are perpendicular to the same plane they must be parallel.

And the converse:

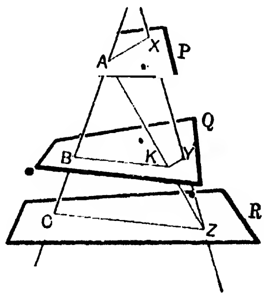
If one line is perpendicular to a plane, then any other straight line, which is parallel to it, must also be perpendicular to the plane.

[Vertical lines in a room, with the horizontal floor as the plane, afford very familiar examples. Especially note § **10** on page 444.]

PROPOSITION 42.

§ 8. **General Enunciation.** *Straight lines which are cut by parallel planes are cut proportionally.*

Particular Enunciation. ABC and XYZ are the lines; **P**, **Q** and **R** are the planes. To prove AB : BC = XY : YZ.



Construction. Join AZ cutting the plane **Q** at K. Join AX, BK, KY and CZ.

Proof. ACZ is a triangle, and hence its three sides are in one plane. Also the plane **Q** is parallel to the plane **R**.

∴ the lines BK and CZ are parallel. (Prop. **41.**)

Hence, in the △ACZ, AB : BC = AK : KZ.

Similarly, in the △ZAX, XY : YZ = AK : KZ;

∴ AB : BC = XY : YZ.

Q.E.D.

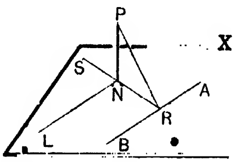
PROPOSITION 43.

§ 9. **General Enunciation.** *To draw a perpendicular to a given plane from a given external point.*

Particular Enunciation. **X** is the plane and P is the point. To construct a perpendicular from P to **X**.

Construction. Draw *any* line AB in the plane **X**. Draw PR perpendicular to the line AB. In the plane **X** draw the line RS perpendicular to AB. Draw PN perpendicular to the line RS. Then PN is the perpendicular required.

(Also the straight line NL, parallel to AB, and through N, is needed in the proof.)



Proof. The straight line AB is, by construction, perpendicular to the two lines PR and RS, at their point of intersection, and so is perpendicular to the plane **PRS**.*

But NL is parallel to AB;

∴ NL is perpendicular to the plane **PRS**.

Hence PN is perpendicular to two lines, namely RS and NL, at their point of intersection; and RS and NL are both in the one plane **X**.

∴ PN is perpendicular to the plane **X**.*

Q.E.F.

N.B.—To draw a perpendicular to a plane from a given point *in* the plane, it is only necessary to draw a parallel to a perpendicular to the plane (from *any* external point).

* If the truth of this is not assumed reference should be made to Proposition V in Appendix II. on page 578.

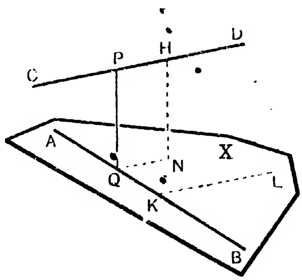
§ 10. If a straight line is perpendicular to a plane it is perpendicular to every straight line in that plane. Some consideration should make this clear. If you think of the junction of two walls of a room as the line, and of the floor of the room as the plane, it is very obvious that the junction is perpendicular to any chalk line you may draw on the floor which goes *through* the corner; and then recollect that, if the line on the floor is *away* from the corner (and, when produced, perhaps does not go through it), for the inclination it is only necessary to draw a parallel to the same *through* the corner. Remember that the angle between two lines is simply the angle between their *directions*.

[Since a plane can be determined in several ways, this important fact conceivably might be overlooked.]

PROPOSITION 44.

§ 11. General Enunciation. *To draw a straight line perpendicular to each of two given straight lines (not in the same plane).*

Particular Enunciation. AB and CD are the two straight lines. To draw a straight line, which is perpendicular to each of them.



Construction. Take *any* point K on AB. Through K draw the straight line KL parallel to CD.

Let **X** be the plane through AB and KL.

Take *any* point H on CD.

From H draw HN perpendicular to the plane **X**. (This is done by the last proposition.)

Through N draw NQ parallel to LK (necessarily in the plane **X**) to meet AB in Q.

From Q draw QP parallel to NH to meet CD at P. (This line will necessarily meet CD, for CD, HN and QN are in one plane.)

Proof. The opposite sides of the quadrilateral PHNQ are parallel, by construction, and HN (which is perpendicular to the plane **X**) is perpendicular to NQ (a line in the plane **X**).

So that PHNQ is a rectangle.

Thus PQ is perpendicular to both AB and CD.

Hence PQ is the line required.

Q.E.F.

N.B.—The common perpendicular PQ *is the shortest distance between the lines.*

For take *any* two points, one on each of the lines. [K and H, in the preceding figure, will do well, for they were *any* points on the two lines.]

Now HN, being perpendicular to the plane **X**, is shorter than any oblique line (*e.g.* HK) to the plane. [HK might be joined in imagination only, for it is not wanted in Prop. **44**.]

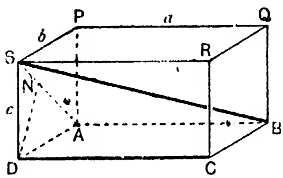
But HN = PQ.

Hence PQ represents the shortest distance between the lines.

§ **12.** As an **Example** of calculation consider the following:

ABCD is the floor of a room (AB = CD = a and BC = DA = b), also PQRS is the ceiling of the room, which is c high. Find the distance between the lines SB and DC.

[The essential thing is to get a plane through one line and parallel to the other line (and, for working purposes, take those in the most convenient positions), then to work with the length of a perpendicular from a point in the line to that plane (theoretically *any* point, but in practice the *most convenient*, for the perpendiculars are necessarily all equal).]



Now AB meets SB and is parallel to DC, so (for convenience) take **SAB** the plane and D the point. Draw DN perpendicular to the plane **SAB**.

[DN will, necessarily, be in the *plane* triangle DSA, and work with that only.]

$$\text{DN} \times \text{SA} = \text{twice the area of } \triangle\text{DSA} = \text{DA} \times \text{SD};$$

$$\therefore \text{ DN} \times \sqrt{b^2+c^2} = bc;$$

$$\therefore \text{ DN} = \frac{bc}{\sqrt{b^2+c^2}}.$$

So that the distance between SB and DC is

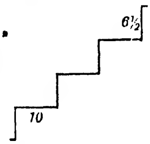
$$\frac{bc}{\sqrt{b^2+c^2}}.$$

N.B.—To find the actual line, a parallel to DC (from N) is needed. If this meets SB at K, then the actual line is from K parallel to ND.

EXAMPLES 41 a (CALCULATIONS. SPECIAL CASES).

O **1.** A cake is cut, along 12 radii, into 12 equal slices. At what angle are the plane faces of a slice inclined?

2. PA, QB, RC and SD are 4 parallel edges of a cube PQRSABCD, of which PQRS and ABCD are opposite faces. Sketch the cube. What is the inclination of the edges PQ and BC?



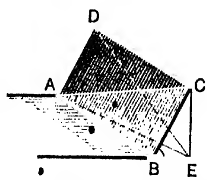
3. A straight flight of stairs has a $6\frac{1}{2}''$ rise and $10''$ tread. At what angle are its banisters inclined to the horizontal? A tea-tray slides down the stairs; at what angle is the plane of the tea-tray inclined to the horizontal?

4. A ladder, whose foot is 4 ft from a wall, just reaches up to a window 20 ft. above the ground. At what inclination is it to the horizontal plane? It is now turned against another wall 6 ft. from the foot of the ladder. How high up will it reach, and what is its inclination then?

5. Two points are taken, on the floor of a room, respectively 3 and 4 feet from a corner of the room and on the two walls (which meet at right angles at that corner). A third point is taken, 2 feet above the floor, on the junction of those two walls. What is the inclination, to the floor, of the plane containing those three points?

6. A scaffolding pole leans in the angle of two walls at right angles. The pole is 30 ft. long and its foot is 8 ft. from one wall and 6 ft. from the other wall. What is the inclination of the pole to the ground?

7. A walking-stick, $36\frac{3}{4}''$ long, is leaning in an angle of a room against the junction of two walls, which are at right angles. Its foot is $16''$ from one wall and $12''$ from the other. What is the inclination of the stick to the ground? Also, what is its inclination to the vertical?



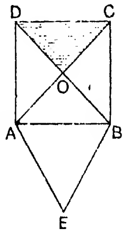
8. A flat rectangular board ABCD (AB $= 20''$, BC $= 16''$) is placed on a horizontal plane, and the edge CD is raised until it is $13''$ above the plane, AB remaining in contact with the plane. CE is the perpendicular from C on to the plane. Calculate the distances AC, BE, AE, DE to the nearest tenth of an inch, and find the angles which AC and BC make with the horizontal plane.

9. B and Q, in a horizontal plane, are the feet of two vertical poles AB, 14 ft. high, and PQ, 27 ft. high. The positions of B and Q are given in the accompanying table:

	Northing.	Easting.
B	5 ft.	3 ft.
Q	26 ft.	23 ft.

What is the inclination of the line AP?

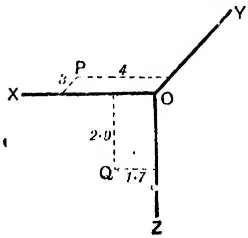
10. The figure shows a square ABCD and an equilateral triangle ABE on opposite sides of a common base AB. The diagonals BD and AC of the square cross at O, and the triangle DOC is cut away. Suppose that the triangles AOD and BOC are turned about AO and BO respectively, each through a right angle, thus bringing OD and OC together at right angles to the plane AOB, which is supposed horizontal. In this way the points C and D will coincide at a position F situated vertically over the point O. Then suppose that the triangle ABE is turned about AB until AE and BE lie along AF and BF respectively. Determine by drawing and measurement the number of degrees through which the triangle ABE must be turned round AB.



* Denote the length of a side of the original square by a, and find in terms of a an expression for the volume of the pyramid formed.

11. A room is 9 ft. high. An electric-light switch is 4 ft. above the floor. A wire goes from switch to bulb thus: Vertically up to the junction of the-wall-on-which-it-is with the ceiling, thence for 1 ft. along that junction, now (at right angles) on the ceiling for 6 ft. to a point from which it hangs for 3 ft. What is the inclination, to the floor, of the direct line from switch to bulb?

12. P and Q are points on two faces of a brick which meet along OX. Their positions are given by their coordinates on the figure. Find the inclination (1) of the line PQ to the face XOY, and (2) of the same line to the line OX.

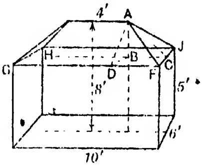


13. A set-square, with angles 30°, 60° and 90°, rests (with its hypotenuse on a horizontal table) against a prop, so that the plane of the set-square is inclined at an angle of 45° to the horizontal. At what angles do its other two edges slope ?

* **14.** A picture frame is 24″ by 20″. It is standing on the floor in an artist's studio and leaning against a wall, so that its plane is inclined at 70° to the horizontal. Find the inclination of its diagonals to the horizontal (1) when a 24″ side is on the floor, and (2) when a 20″ side is on the floor.

* **15.** The sloping front of a bureau is 3 ft. long and 16 in. wide, (open it forms a flap on which to write). When the bureau is shut, this is inclined at 45° with the horizontal. At what angle are its diagonals inclined to the horizontal ?

* **16.** The figure shows a symmetrical glass skylight, with four vertical rectangular panes, two sloping triangular panes and two sloping panes of trapezium shape. The dimensions are shown in the figure.

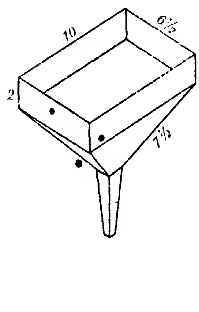


From A is drawn AC perpendicular to FJ, AD perpendicular to GF and AB perpendicular to the centre line HC. What is the length of BC ? of BD ?

At what angle are the sloping planes inclined to the horizontal ?

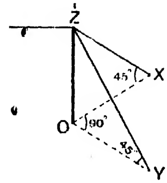
* **17.** A pyramid (on a square horizontal base with sides 10 cm.) has its sloping edges each 13 cm. At what angles are (1) its sloping faces and (2) its sloping edges inclined to the base ?

* **18.** A funnel, for guiding petrol into the tank of a motor car, has a mouth in the shape of a rectangle (10″ by $6\frac{1}{2}$″). After the 2″ portion, the four sides slope down to a central square (1″ each way) to which the nozzle is attached. The lengths of the junctions of consecutive sloping planes are $7\frac{1}{2}$″.



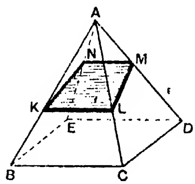
(1) At what angles are the planes of the four sloping sides inclined to the rectangle ?

(2) At what angles are the lines forming the junction of those planes inclined to the same rectangle ?



* **19.** A post, OZ, is kept vertical by two guy ropes, ZX and ZY, which are inclined at 45° to the horizontal, and also a horizontal rope at Z. The horizontal lines, OX and OY, are at right angles. What is the inclination of the plane ZXY to the horizontal ?

* **20.** A-BCDE represents a wooden pyramid on a square base. Each edge of the pyramid is 12 cm., so that each of the triangular faces meeting at the apex A is an equilateral triangle of side 12 cm.



Imagine points K, L, M, N taken along the edges running from A, such that AK and AL are each 8 cm., while AM and AN are each 4 cm. By cutting away the top part of the pyramid, so that the edge of the saw passed through K, L, M, N, the section KLMN shown shaded would be obtained. Find the lengths of the sides of this section either by drawing the equilateral triangles full size, or by calculation, making your method clear; then give a full-size drawing of KLMN. How does the symmetry of the figure help you ?

* **21.** A pyramid stands symmetrically on its base, which is a regular pentagon of side 3 inches. The height of the pyramid is 7 inches. Find the length of one of the sloping edges and the angle between a sloping face and the base.

* **22.** The sides of a triangle ABC (which is horizontal) are $a = 2100$ ft., $b = 1700$ ft. and $c = 1000$ ft. A plane bed of rock comes to the surface in the line BC. A shaft, sunk at A, strikes the bed of rock at a depth of 291 ft. What is the inclination of the bed of rock to the horizontal ?

* **23.** A line of trenches runs due north and south, and an aviator to the east of the line at a height of 1000 feet observes the angle of depression of the nearest point in the trenches to be 26°, when his engine fails to work. If he glides down at an angle of 10° to the horizon and requires to land on the line of trenches south of his present position, determine the direction of the point of the compass for which he must steer

(in degrees south of west) and the distance that his landing place is south of his present position.

[To the nearest degree is quite as accurate as would be possible in practice.]

* **24.** ABC is a board, in the shape of an isosceles triangle, in which $CA = CB$ and $\angle ACB = 80°$. It rests, in the corner of a room, with its sides CA and CB in contact with the walls of the room, and with its base AB on the floor of the room. At what angle is the board sloped to the floor?

* **25.** A small trellis-work gate, made with iron rods, (of the same pattern as the fence in Question **33**), is off its hinges. It is waiting for a coat of paint, and meanwhile is leaning up against a wall. It is inclined at 20° to the *wall*. At what angle are the iron rods inclined to the wall?

* **26.** A straight fence runs east and west. All its posts lean towards the south-east, making an angle of 75° with the (horizontal) ground. Many straight wires, composing the fence, go from post to post. What is the inclination to the ground of the plane containing those wires?

* **27.** In making a sock three knitting needles form a plane, and are held so that two needles are at right angles. The fourth needle is inclined to one of these two needles at 45° and to the other at 60°. At what angle is this fourth needle inclined to the plane of the three needles?

[HINT. Call the 2 needles at right angles OX and OY, and the fourth needle OP. If OZ is at right angles to both OX and OY, find the inclination of OP to OZ. Then, etc.]

* **28.** ABC is a horizontal triangle and AK is one of its medians. M is the middle point of AK. Shafts at A, B and C meet a certain plane bed of rock at depths 200 ft., 380 ft. and 420 ft. respectively. At what depth, below M, is the bed of rock?

* **29.** Shafts are sunk at A, B and C (the corners of a horizontal equilateral triangle whose sides are 450 ft.), and a bed of rock (whose surface is plane) is struck at depths 600 ft., 700 ft. and 800 ft. respectively. What is the inclination of the bed of rock to the horizontal?

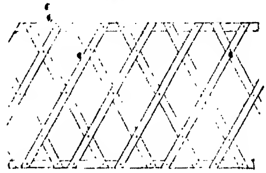
* **30.** A pyramid is formed from a corner of a cube by a plane through the diagonals of the 3 faces which meet at that corner, taking the diagonals which do *not* pass through that corner. It is thus a pyramid on an equilateral triangle as base, and with equal right-angled isosceles triangles as its side faces,

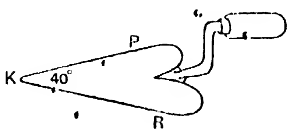
all the right angles being at the common vertex. At what angle are the edges inclined to the base, and at what angle are those three faces inclined to the base ?

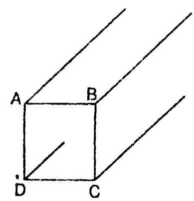
* **31.** The steps of a spiral staircase are 3 ft. wide, and they go round a central column of 1 ft. diameter. At the centre of each step the tread is 11″; in other positions, on the step, the tread is proportional to the distance from the centre of the central column. The rise of each step is 6″. Find the inclination of the stairs (1) at the centre of each step, (2) on the extreme inside, and (3) on the extreme outside.

** **32.** A book is opened so that a 45° angle of a set-square just fits in (the plane of the set-square being perpendicular to the leaves of the book). Suppose that the set-square issues from the pages of the book at points A and B, and meets the hinge line at K. A 30° angle of a different set-square is now put in the book instead (the book being opened the same amount as before). Its plane is inclined to the plane of the 45° set-square. It passes through A and B, and meets the hinge line at Q. If KA = KB = 4″, how long is KQ ?

** **33.** A trellis-work fence, after the accompanying pattern (in which the sloping bars are inclined at 60° to the horizontal bars), is leaning against a vertical wall so that the plane of the trellis work is inclined at 45° to the ground. At what angle are the sloping bars inclined to the ground ?







** **34.** A perfectly flat trowel (rather like a bricklayer's trowel), is pointed to 40°, and has straight edges. It is too wide to go completely into the square hole ABCD with sides 2″.

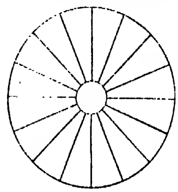
However, it is thrust in as far as it will go, with KP and KR equal and going through A and C respectively. Then an attempt is made to reach down the D-edge with it. How far down the D-edge will it reach?

* * **35.** A fisherman's net is stretched so that all the meshes are square. It is out to dry, and a portion of it is plane. The set of strings parallel to one side of the square are inclined at 30° to the ground, while the other set is inclined at 45° to the ground. At what angle is the plane inclined?

* * **36.** A plane bed of rock strikes N.E. and S.W., and its dip is 30° (towards the N.W.). [The *strike* of a bed is the direction of a horizontal line on the plane of the bed, and the *dip* of the bed is its inclination from the horizontal.]

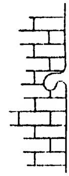
103 yards N. of a point where the bed comes to the surface a shaft is sunk. At what depth will it meet the bed of rock? The surface is horizontal.

Plan of spiral staircase.



* * **37.** A spiral staircase goes round a circular column whose diameter is 1 foot. The width of each step is 3 feet. There are 16 steps in the complete circuit. The steps are each 7 inches high. On the inside of the stairs is hung a rope, round the column, as a hand grip. On the outside of the stairs a handrail is incised in the wall (see figure). What are the respective inclinations of these two? [It is to be assumed that the rope is stretched tight, and not hung loosely to form a doubtless more convenient grip.]

Vertical section showing handrail in wall.

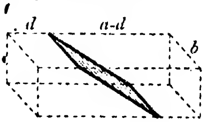


* * **38.** ABC is a triangle, on horizontal ground, with BC 510 yds., CA 550 yds. and AB 260 yds. Shafts at A, B and C meet a plane bed of rock, below the surface, at depths 50 ft., 638 ft. and 1205 ft. respectively. Show that the inclination of the bed of rock is 41° (to the nearest degree).

EXAMPLES 41 b (CALCULATIONS. GENERAL CASES).

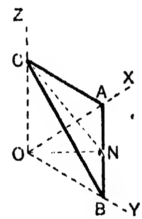
1. Considering the figure on page 446, what is the distance between the following pairs of parallel lines, (1) PQ and DC, (2) SP and CB, (3) SD and QB ?

2. A rectangular parallelepiped (a by b by c) is divided into 2 equal portions by a plane, shaded in the diagram. The top line of that plane is d from one end, while the bottom line of that plane is d from the other end. What is the inclination of the shaded plane ?



3. At what angle is the diagonal of a cube inclined (1) to the faces of a cube, and (2) to those edges of a cube through whose junction it passes ?

4. An equilateral triangular board, ABC, is placed in the corner of a room, so that its angles are on the edges OX, OY and OZ. XOY is the floor. OZ is vertical. (OA = OB = OC necessarily.) At what angle is the board ABC inclined to the floor ?



[HINT. To find this angle (as a matter of fact ONC), the lines ON and CN must *both* be at right angles to AB. (Consider a (vertical) plane revolving round OZ until it is at right angles to AB. N is the point where this plane now cuts AB. AB is at right angles to the straight lines NO and NC (as it is at right angles to the plane NOC). Let OA = OB = OC = k. Calculate ON. Then, etc.]

* **5.** What is the distance between the edge of a cube and a diagonal of the cube which never meets it ? The cube has edges a.

* **6.** ABC is a triangle on horizontal ground. Shafts at its three corners strike coal at depths h_1, h_2 and h_3 respectively. If the coal is in a plane bed and if a shaft is sunk at G, the centre of gravity of the triangle ABC, at what depth, in that shaft, will coal be struck ?

* **7.** ABCD is a parallelogram on horizontal ground. Shafts at the 3 corners A, B and C meet a bed of rock at depths h_1, h_2 and h_3 respectively. If the bed of rock is plane, at what depth will a shaft at D meet it ?

* **8.** A plate rests on a horizontal table. Its circular rim has a diameter $2R$ and is h above the table. A vertical napkin ring (with diameter $2r$) is standing on the table, and is pressed against the rim of the plate. What is the maximum penetration of the rim of the plate into the napkin ring in these circumstances?

* **9.** A room is a long, b wide and c high. PQRS is its ceiling and ABCD its floor (the letters corresponding). The shortest distance between the diagonal SB and the side DC has been worked out on page 446 (but you should consult the figure on that page for the letters). F and G are points on SB and DC respectively, such that $SF = \frac{1}{3}SB$ and $DG = \frac{1}{3}DC$. Calculate the length of FG.

* * **10.** The length, breadth and height of a room are a, b and c. Two straight strings are stretched, the former being a diagonal of the room, and the latter being a diagonal of its floor (but *not* through the same corner as the former string). What is the angle between the two strings?

* * **11.** A room is a long, b wide and c high. One string is stretched tight as a diagonal of the room. Another string is along one of the long sides of the floor. The strings do not meet. [The strings are SB and DC in the figure on page 446]. If the shortest distance between the strings is along a line KL (K being on the former and L on the latter), how long is DL?

* * **12.** A pyramid VABC is on an equilateral triangular base ABC of edge a. The other 3 edges are each of length l. PQRABC is the prism, on the same base ABC, and of the same height (P being vertically above A, Q above B and R above C). Calculate the length of the perpendicular from the middle point of the line QR to the plane VBC.

[Hint. If M is the middle point of the line QR and K is the middle point of the line BC work with the rectangle PMKA.]

* * **13.** A box is a by b by c. Show that the sum of the squares of the cosines of the inclinations of one of its diagonals to its 3 faces is twice the sum of the squares of the cosines of the inclinations of the same to its three edges.

* * **14.** A door, a high and b wide, is ajar at an angle of 2θ. At what angle is a diagonal of the open door inclined to the *other* diagonal of the same door, when shut?

[Hint. A question, resembling this, but with *same*

instead of *other*, was proposed in Examples 40 a 21. Keep the diagonal of the shut door unchanged, but then imagine the hinges on the opposite side and a phantom door ajar to the supplementary angle.]

* * **15.** A square piece of stiff paper CDEF (Fig. 1) is folded along the line AB, where A and B are the middle points of opposite sides. It is opened out till the two halves are inclined at an angle θ, and then placed on a desk which slopes at an angle α to the horizontal, in such a manner that the plane ABDC lies on the desk and the crease AB lies along a line of greatest slope of the desk (Fig. 2). Denoting a side of the square by x, express in terms of x, θ and α the vertical heights of the points A and E above the point B. Hence find the sine of the angle which the line AE makes with the horizontal plane. Verify that AE is horizontal if $\sin\theta = 2\tan\alpha$.

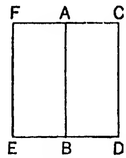


Fig. 1

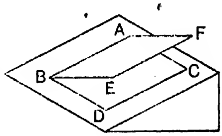
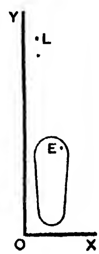


Fig. 2

* * **16.** The coordinates of the point P are a, b, c. The coordinates of the point Q are f, g, h. YOZ and ZOX are two plane mirrors. A ray of light issues from P, and travels to Q after reflection in the two mirrors. What is the length of the path of the light?

* * **17.** In the previous example, if all three (YOZ, ZOX and XOY) were plane mirrors, and if light went from P to Q, after successive reflection in all three, what is the length of the course?

* * **18.** One is sitting in a bath near a corner of a room. The floor-plan is given. The surface of the water is at a height k above the floor. One's eyes are vertically over E, whose x and y coordinates are p and q, and at a height r above the water. An electric light is vertically above L, whose x and y coordinates are a and b. There are vertical mirrors hanging on the walls OX and OY, in such positions that one sees, at a point on the water surface, distant horizontally



f from E, the reflection of the light, reflected from the mirrors and the water. What is the height of the electric light above the floor ?

EXAMPLES 41 c (RIDERS).

O **1.** What is the locus in space of a point at a constant distance from a fixed straight line ?

O **2.** What is the locus in space of a point equidistant from two given points ?

O **3.** What is the locus of points equidistant from two given planes ?

O **4.** What is the locus of points equidistant from three given planes ?

O **5.** How many points are there which are equidistant from four given planes ?

O **6.** Through a given point to draw a plane parallel to a given plane. What is the construction for this problem ?

O **7.** How is a straight line drawn, through a given point in space, to intersect two given straight lines (which are not in the same plane) ?

O **8.** From a given point, perpendiculars are drawn to a plane, which passes through a given straight line. What is the locus of the feet of the perpendiculars ?

O **9.** What is the locus of a point equidistant from three given concurrent lines not in a plane ?

10. The sum of the distances of two points from a plane is twice the distance of their mid point from the plane.

11. Show that the angle between 2 planes is the *greatest* angle between two lines in them, which issue from any point in their common section.

12. Show that the angle between a line and a plane is the *least* angle that the line makes with any line in the plane through the intersection.

13. If the edges, which meet at the vertex of a pyramid, are of different lengths, the longest of these makes the smallest angle with the base.

14. Find a point in a given plane such that the sum of its distances from two given points (not in the plane but on the same side of it) may be a minimum.

15. P is any point on the circumference of a given circle, and O is a fixed point outside the plane of the circle. OP is cut in a given ratio at Q. Prove that, for different positions of P, the locus of Q is a circle.

16. If PQ and XY are two parallel straight lines, then the intersection of any two planes, through PQ and XY respectively, is parallel to both PQ and XY.

17. From a given point P, outside a plane **X**, a perpendicular PN is drawn to the plane **X**; and another perpendicular PR is drawn to *any* line AB in the plane **X**; prove that NR is perpendicular to AB.

18. If a straight line is parallel to a plane, show that any plane passing through the given straight line will intersect the given plane in a line of section which is parallel to the given line.

19. Equal straight lines are drawn to meet a plane **X**, from a point P, outside it. Prove that the equal straight lines are equally inclined to the plane **X**.

[HINT. From P draw PN perpendicular to the plane **X**.]

20. Through a given point a number of lines are drawn making a given angle with a given plane. What kind of surface do such lines form? Draw sketches showing the general appearance of the surface, when the given point is on the plane, and when it is not on the plane. In the latter case indicate and state the nature of the locus of the intersections of the lines with the plane.

*** 21.** The locus of a point, the difference of the squares of whose distances from two given points is constant, is a plane.

*** 22.** If a point be equidistant from the vertices of a right-angled triangle, its join to the mid point of the hypotenuse is perpendicular to the plane of the triangle.

*** 23.** If S is the centre of the circle circumscribed about the triangle ABC, and if SP is drawn perpendicular to the plane of the triangle, show that any point in SP is equidistant from the vertices of the triangle.

* 24. A, B and C are points on the lines OX, OY and OZ, which are mutually at right angles. $OA=a$, $OB=b$ and $OC=c$. Prove that the area of the triangle ABC is $\frac{1}{2}\sqrt{b^2c^2+c^2a^2+a^2b^2}$.

[HINT. Let a plane revolve about OZ until it is at right angles to BA. Let it cut BA at K. Calculate OK in the triangle OBA, and then BK. Now find CK in the triangle CBK, etc.]

N.B. If Δ_O is the area of the triangle ABC opposite to O, etc.; you should notice that this is equivalent to

$$\Delta_O^2=\Delta_A^2+\Delta_B^2+\Delta_C^2.$$

This is an extension of Pythagoras' Theorem.

* 25. ABC is any triangle and G is its centre of gravity. V is any point in space. Prove that $VA+VB+VC>3VG$.

[HINT. Bisect BC at K. Prove $VB+VC>2VK$ by completing a parallelogram, two of whose sides are VB and VC. Now draw the VKGA section. Recollect that KG is $\frac{1}{3}$KA. Prove $VA+2VK>3VG$.]

* * 26. Draw two parallel planes, one through each of two straight lines, which do not intersect and are not parallel.

[HINT. First draw the straight line which is perpendicular to each of the given straight lines.]

* * 27. An acute-angled triangle ABC is supposed to be cut out of paper and divided into four triangles by joining up D, E, F, the middle points of the sides BC, CA, AB, respectively. The triangle AEF is gradually folded over about the side EF. Prove that the projection of its vertex A on the plane of the triangle DEF moves along a straight line perpendicular to EF. If now the triangles BFD and CDE are similarly folded about FD and DE till the three can be joined together so as to make a triangular pyramid or tetrahedron, show how to find the projection of the vertex on the base DEF.

Supposing the sides of the triangle ABC to be 5, 6 and 7 inches long respectively, draw a figure of the triangles before folding, and on it construct the projection of the pyramid on the base DEF. Determine the altitude of the pyramid.

* * 28. ABC is *any* triangle. $A_1B_1C_1$ and $A_2B_2C_2$ are two distinct, quite general, positions of the same in space. (*N.B.* The planes $A_1B_1C_1$ and $A_2B_2C_2$ cannot be parallel, for the positions are quite general.) It is necessary to transfer the triangle

ABC from position 1 to position 2 *by rotation about a single certain point.* How is the correct position of that point to be determined?

What would be the difficulty if the planes $A_1B_1C_1$ and $A_2B_2C_2$ were parallel?

What happens, in the special case, if $B_1B_2C_2C_1$ were a rectangle, even though the planes $A_1B_1C_1$ and $A_2B_2C_2$ were not parallel?

* * **29.** What is the locus of the middle point of a straight line of constant length whose extremities lie one on each of two non-intersecting straight lines, having directions at right angles to one another?

* * **30.** If AB, BC, CD are straight lines not all in one plane, show that a plane which passes through the middle point of each one of them is parallel both to AC and BD.

* * **31.** The middle points of the six edges of a cube, which do not meet a particular diagonal, all lie in a plane perpendicular to that diagonal.

* * **32.** Two right-angled triangles BAC, ABD (A and B being respectively the right angles), are situated in planes perpendicular to each other: show that the join of the mid points of AB and CD is perpendicular to AB.

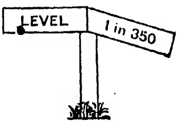
[It may help you to draw the 2 $\triangle$s correctly on paper with a common side AB, to cut out the quadrilateral ACBD and then to bend it across AB at right angles, and to use the model.]

* * **33.** A tetrahedron has six dihedral angles. Prove that the planes which bisect those angles must pass through a point. What is that point called?

CHAPTER XLII.

GRADIENTS.

§ 1. For **railway lines** are the **gradients**, as given by the boards at the sides of the line, to be reckoned as 1 in . . . *horizontal* or 1 in . . . *along the slope?* In Trigonometry we should say, "Is it a question of tangents or sines?"



Consider the gradient 1 in 37, which is the steepest which occurs in any *long* run* (in two miles) in Great Britain. In the first place the 37 is "rounded off," and, in extreme cases, can be anything between $36\frac{1}{2}$ and $37\frac{1}{2}$. This makes a possible difference in vertical height of nearly 8 *feet* in those two miles. In the second place the difference, as to whether tangents or sines are meant, is about $1\frac{1}{4}$ *inches*. Thus, for the steepest railway gradients, the question of "rounding off" far outweighs the question of "horizontal" or "along the slope"; *and so it does not matter, in practice, for these boards.* If distances are measured on the ground (and therefore, necessarily, along the slope) sines are correct; but if distances are measured on the construction plans, they would be horizontal, and tangents are right.

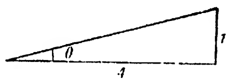
It is necessary that no false impression should be obtained from the above. What is good enough for the boards at the sides of the lines is not good enough for the railway dynamics; in them gradients are invariably measured by sines.

* Steeper gradients for *short* lengths occur, but they are not common, and are "rushed."

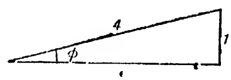
§ 2. For the gradient of 1 in 5 (a *very* severe hill for roads) the angles are 11° 19′ (−) and 11° 32′ (+), and even then there is a trivial difference between tangents and sines. This is entirely disregarding the fact that the 5 itself (in practice) is often "rounded off," and a far greater variation is perhaps already swallowed.

For large angles, it must always be specifically stated whether "horizontal" or "along the slope" is meant.

1 in 4 (horizontal); $\tan\theta = \frac{1}{4}$, whence (from tables) θ = about 14°.



1 in 4 (along the slope); $\sin\phi = \frac{1}{4}$, whence (from tables) ϕ = about 14½°.

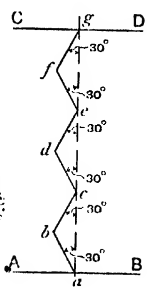


EXAMPLES 42 (CALCULATIONS. SPECIAL CASES).

1. The steepest slope of a uniform hillside is 1 in 2 (1 vertical, 2 horizontal). A path, inclined to the line of greatest slope, is to be made up it, at a gradient 1 in 7. Find the inclination of the path to the line of greatest slope.

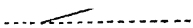
2. A hill surface may be assumed to be a plane with a uniform slope of 25° to the horizontal. You climb the hill by following a path, the centre line of which makes an angle of 20° with the line of greatest slope of the hill surface. Find how many feet you rise vertically when you have walked 1000 feet along the path.

3. The figure represents the side of a hill, which has a uniform slope upwards from AB to CD of 18 degrees to the horizontal. Starting from a point a at the foot of the hill, a zigzag path a, b, c, d, e, f, g is constructed up the hill in the manner shown in the figure; the finishing point g is 600 ft. higher in level than the starting point a, and ag is a line of greatest slope of the hill. Find the length of the zigzag path a, b, c, d, e, f, g, and the gradient of the path.



* **4.** A uniform hillside slopes at a gradient of 1 in 3 (1 vertical, 3 horizontal). What is this angle to the nearest half degree? If a path is made up the hillside, zig-zagging, each lap being inclined to the line of greatest slope at an angle of 67°, what is the gradient of the path?

* **5.** The gradient of a path is the ratio of the height gained to the horizontal distance traversed in gaining that height. Prick off the figure, and find the gradient of each portion of the path shown in the figure; the dotted line is horizontal.



Measure also the angle which each portion of the path makes with the horizontal.

The slope of a path is sometimes taken to mean the gradient as defined above, and sometimes the ratio of the height gained to the distance traversed *along the slope* in gaining that height. Find for each portion the slope in this second meaning and compare with the gradient.

* **6.** When a man walks along a sloping path, the distance he traverses horizontally is less than the distance on the slant. The slope of a path is often stated as "1 in N," the N being measured sometimes along the slant and sometimes horizontally. On each method of taking N draw a slope of 1 in 2 and measure the angle of slope.

For a slope of 1 in 10 see if you can distinguish by drawing between the angles of slope as given by the two methods of taking N.

Of two roads, one sloping at 1 in 13 and the other 1 in 50, which is steeper? Show them on squared paper with sufficient accuracy to justify your answer.

* **7.** Cut a sheet of paper into some such shape as ACDE (Fig. 1). Then fold AB at right angles to BC, and place the paper before you on the desk or table (supposed horizontal) with AB, BC on the desk (Fig. 2).

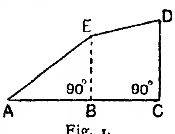


Fig. 1.

Imagine now that AED (Fig. 2) represents a path on a hill-side, the slope AE having a vertical rise of 1 ft. for every 5 ft. horizontal and ED a slope in which there is a rise of 1 ft. in every 12. If AB is a mile and BC three-quarters of a

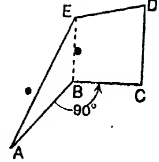


Fig. 2.

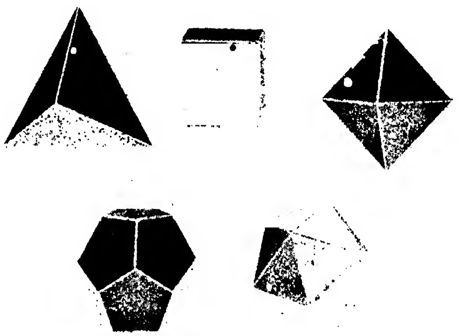
mile, find how many feet E is higher than A, and D than E; also calculate the length of AC.

If D were 1500 ft. higher than A and AC the same length as before, what would be the average rise in passing direct from A to D? Give the result in the form 1 ft. vertical to x ft. horizontal.

CHAPTER XLIII.

REGULAR SOLIDS.

§ 1. There are five (and only five) regular solids (*i.e. convex solids with their faces equal regular polygons*). [*N.B.* —If a solid is entirely on one side of any face it is said to be "convex."]



These are

(1) **Regular Tetrahedron :**

4 faces—Equilateral triangles.

(2) **Cube :**

6 faces—Squares.

(3) **Regular Octahedron :**

8 faces—Equilateral Triangles.

(4) **Regular Dodecahedron :**

12 faces—Regular Pentagons.

(5) **Regular Icosahedron :**

20 faces—Equilateral Triangles.

These regular solids are known as the "Platonic solids," for they were extensively studied by the Platonists.* **Plato** (about 400 B.C.) was a pupil and friend of Socrates, and sought in Arithmetic and Geometry a key to the Universe. When questioned about the occupation of the Deity, Plato answered, "He geometrises continually." He put a great value on the study of mathematics as a necessity for higher speculation; and the inscription over his porch, "Let no one who is unacquainted with geometry enter here," shows the importance he attached to the subject.

If a solid is regular, or not, it is often called a **Polyhedron**.

§ 2. Proof that only 5 regular polyhedra are possible.

Now the size of each exterior angle of a regular n-agon is $\frac{360^\circ}{n}$.

So for 3, 4, 5, 6, ... etc. sides the angles are 120°, 90°, 72°, 60°, ... etc., *decreasing*.

∴ the interior angles (the supplements) are 60°, 90°, 108° 120°, ... etc., *increasing*.

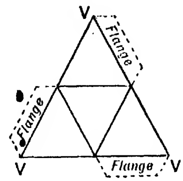
All the plane angles, at a corner of any polyhedron, must not "fill up" the point, and so must total less than 360°.

Now	Three times 60°	is less than 360°		(regular tetrahedron);
	Three times 90°	,,	,,	(cube);
	Three times 108°	,,	,,	(regular dodecahedron);
	Four times 60°	,,	,,	(regular octahedron);
	Five times 60°	,,	,,	(regular icosahedron).

So that these 5 regular polyhedra are possible, and no others, for the next (and all others) to be considered would make all the plane angles at a point not less than 360°.

* Much of this note is from *History of Mathematics*, by Cajori.

§ 3. On a stiff piece of paper you can draw the accompanying figure, of 4 equilateral triangles, with flanges on 3 edges.



You can cut it out, bend across the edges of the 4 equilateral triangles, gum the flanges, bring the points V together, and so make a good model of a regular tetrahedron. It is no more difficult to devise a figure, composed of 6 squares, to make a model of a cube. The regular octahedron model does not require much ingenuity. The regular dodecahedron and icosahedron models require considerable care. Of course, it is possible to construct models, other than those of regular solids, on this principle. Models of the cells of bees afford good examples, while the forms of crystals give great varieties.

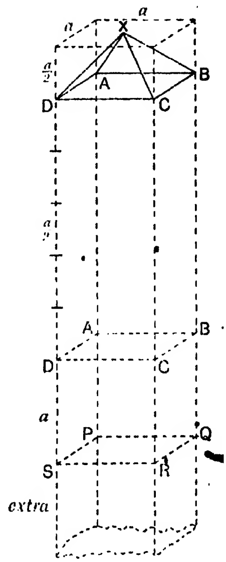
It might be well to note that the models which served for the photograph on page 465 were made in this way. Of course their edges were accentuated in the photograph.

* § 4. The **Rhombic Dodecahedron** is interesting. Though all its twelve faces are identically equal rhombi, they are *not regular figures*, and so this polyhedron is not classed as one of the five. Each face is a rhombus of a special kind. It is called a **semi-regular solid.** It is a common form in crystallography.

There are many ways in which this solid may be conceived. Two ways are treated here, (1) a building-up plan and (2) a cutting-down plan.

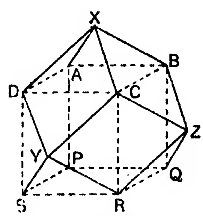
(1) Building-up a Rhombic Dodecahedron.

Instruction for making a model by carpentry follows. Obtain a piece of wood, a prism in shape, with a square cross-section. The length should be well over 4 times the side of the square (extra length to allow for wastage owing to saw-cuts and, especially, for a vice to grip on to). Point the end to a square pyramid with height, centrally, exactly half the side of the square. Cut off the square pyramid. Point the remainder of the wood similarly, and cut off, in succession, the square pyramids (six in all). Then cut off the cube.



Now glue a pyramid on to each of the six faces of the cube, and the model is made.

[Notice that the six pyramids can be fitted (points inwards) to form one cube. This gives the fact that the volume of a rhombic dodecahedron is twice that of *this* cube. Clearly this experiment should be made before any gluing is done.]



On each of the 6 faces of the cube one of these square pyramids (height half of edge of base) is fastened. Only 3 are shown in the figure.

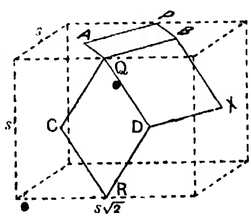
The appearance of the model can be improved by filling up imperfections with putty, and then giving all a coat of paint.

The figure shows a rhombic dodecahedron built up on this plan. The outline of the visible parts of the original cube is accentuated.

The important thing in the proof is that the quadrilaterals (such as DXCY) are plane and not skew.

(2) **Cutting down a prism to leave a rhombic dodecahedron.**

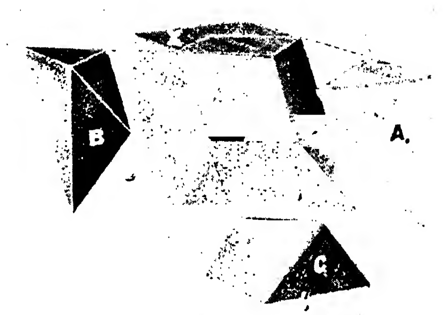
This solid may be obtained as follows. [In making a model by carpentry, it only involves 8 cuts; and the fewness commends the plan.] First a prism, of square section, $s\sqrt{2}$ by s by s is obtained. [Note $s\sqrt{2}$ is equal to the diagonal of the square end.] Secondly, P, Q, R and S (S is not shown in the figure) are the mid points of the long edges. Thirdly, AB, CD, ... etc. (4 in all, though only 2 are shown in the figure) are central lines on the 4 large faces, each equal to $\frac{1}{2}s\sqrt{2}$. Fourthly X and Y (the latter is not shown in the figure) are the mid points of the square ends. That PAQB and QCRD are rhombi is obvious. Now you must prove QDXB plane, and this is most easily done by proving QB and DX parallel; it is simple to prove them equal.



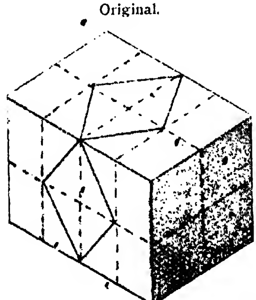
[In carpentry the 4 rhombi on the large faces are left intact. 8 saw-cuts are made, the plane QDXB is one, the other 7 are precisely similar.]

To prove the rhombi equal, prove that their long and short diagonals are equal. What are QX and BD in terms of s?

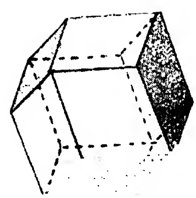
The figure is from a photograph. 3 of the 8 cuts have been made. The pieces A, B and C are cut from the parent block X. The edges have been accentuated.



Also a mathematical drawing of the final solid is given, *but in nearly all calculations, connected with the solid, go back to the original.*



Prism $s\sqrt{2}$ by s by s marked ready for cutting.



Rhombic dodecahedron.

** § 5. Since 3 (and only 3) of the *regular polygons* in *plane* geometry can (*by themselves*) be used as tiles of the same size and pattern to "fill up" areas, [though definite mixtures are possible, *e.g.* an equal number of squares and regular octagons, and the more elaborate pattern illustrated in Volume I. on page 196], one, very naturally, inquires which of the *regular polyhedra* in *solid* geometry will (*by themselves*) **fill up space.**

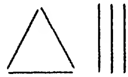
Of the five regular polyhedra, only 1, the cube, can be so used (by itself). It fills up space in layers, and it is easy to imagine all space occupied by equal cubes, like bricks in a wall; there would be no gaps nor overlapping. It is noteworthy that the property of filling up space (*by themselves*) belongs also to Rhombic Dodecahedra; for if equal circles, in a plane, are allowed to arrange themselves naturally with the minimum waste of space, each circle has six neighbours: if they are put under pressure, and flattened against each other, they become regular hexagons. The corresponding thing, for 3 dimensions, is that equal spheres have twelve neighbours each and would be squashed into Rhombic Dodecahedra. A mixture is possible, and if a regular tetrahedron is stuck on to each of two opposite faces of a regular octahedron, all of the same length edge, one parallelepiped is formed (all the 6 faces are equal rhombi with angles 60° and 120°). Now space can be filled up with congruent parallelepipeds, hence space can be filled up with regular tetrahedra and regular octahedra in the ratio of 2 : 1 in number.

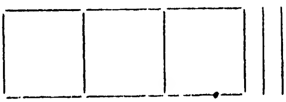
EXAMPLES 43 (MOSTLY GENERAL CASES).

O 1. How many (1) faces, (2) vertices and (3) edges has a cube ?

O 2. How many (1) faces, (2) vertices, and (3) edges has a regular tetrahedron ?

O 3. 3 equal matches are laid on the table to form an equilateral triangle. 3 more equal matches are given with the instruction, "With all 6 matches form 4 equilateral triangles, at the same time, all equal to the original." How is it to be done ?





O 4. 10 equal matches are arranged to form 3 equal squares. 2 equal matches are to be added, to make in all 6 perfect squares exactly like these in size. How is it to be done ?

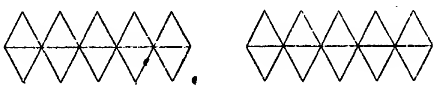
O 5. How many (1) faces, (2) vertices, and (3) edges has a regular octahedron ?

O 6. If the length of an edge of a cube is a, how long is its diagonal ? Give the result in a surd form.

O 7. How many degrees are there in the angles of a regular pentagon ? Notice that 3 of these angles are in all less than 360°. How many regular pentagons are needed to form a regular solid ? How many faces, vertices and edges has it ? What is its name ?

O 8. Two skeleton equilateral pyramids, on square bases, are made of equal matches. How many matches in all make the two separate pyramids, and how many of these can be dispensed with if the pyramids are joined into one regular octahedron ?

N.B.—You might consult the figure on the opposite page.

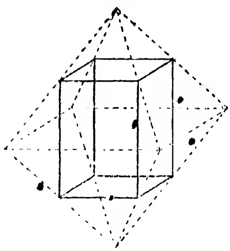


O 9. 50 matches are arranged (in 2 batches of 25 each) to form 20 (10 + 10) equilateral triangles.

These two batches can be bent and fitted to represent a skeleton icosahedron. How many matches can be abstracted ?

O 10. What is the inclination to each other of two opposite edges of a regular tetrahedron ?

11. Find the length of a diagonal of a regular octahedron in terms of an edge a.



12. The mid points of 8 edges (*not all* 12) of a regular octahedron are joined up (as indicated), and form a rectangular prism. If the edges of the octahedron are a, what are the edges of the prism ?

13. Show that the joins of the centres of the faces of a regular octahedron form a cube. If an edge of the octahedron is a, what is the edge of the cube ?

[*N.B.*—The "centre" of a triangle is in general vague, however there is no question as to the meaning if the triangle is equilateral.]

14. Show that the joins of the centres of the faces of a cube form the edges of a regular octahedron. If an edge of the cube is a, what is the edge of the regular octahedron ?

15. You are given a right four-sided pyramid, on a square base, all of whose edges are equal to 5 cm. Draw a figure which when cut and folded in the correct way would exactly cover the pyramid. The figure is a kind of star. Join the four points of the star, and prove that you now get a square. Hence state how you would cut out from a square of side 20 cm. a figure that would fold up to a similarly shaped pyramid; and find the length of each side of this new pyramid, if it is as great as possible.

* **16.** Prove that the shortest distance between two opposite edges of a regular tetrahedron is equal to half the diagonal of the square on an edge.

* **17.** VABC is a regular tetrahedron. M is the mid point of the edge VA, and K is the mid point of the edge BC. If a is the length of the edge of the tetrahedron, what is the length of KM ?

* **18.** Two opposite corners of a regular octahedron are P and Q. A, B, C and D are the corners of the same, forming a square in between them. What is the inclination of the edges AP and QD ?

[HINT. Recollect, for the inclination of two non-intersecting lines, that for one of those lines you should work with *a line through one and parallel to the other*. Now the edge QC is parallel to AP, so consider the inclination of QC to QD.]

* **19.** How far is it from one of the corners of a regular tetrahedron (whose edges are a) to the C.G. of the tetrahedron ?

* **20.** PQRSWXYZ and VABC are a cube and regular tetrahedron, both with unit edge. O is the centre of the cube. The opposite edges VA and BC are bisected at K and M respectively. KM is joined.

(*a*) (1) Draw the section PRYW, including its two lines PY and WR, which meet at O. On the sides of the $\triangle$ROY write their surd values.

(2) Draw the section VMA. On the sides of the $\triangle$VMA write their surd values.

Notice that the angle between diagonals of a cube must be equal to the (dihedral) angle between faces of a regular tetrahedron.

(*b*) (1) Draw the section PRYW, again, and the one line YP, and on the sides of the $\triangle$PWY write their surd values.

(2) Draw the section KMA, and on the sides of that triangle write their surd values.

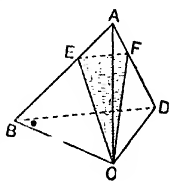
Notice that the diagonal of a cube is inclined to a face of the cube at the same angle as the line joining the mid-points of opposite edges of a regular tetrahedron is inclined to a face of that solid.

* **21.** The centres of the faces of a regular tetrahedron are joined to form a second regular tetrahedron. The centres of the faces of this second are joined to form a third, and so on *ad infinitum*. If V is the volume of the original solid, what is the sum of the volumes of all the solids formed ?

* **22.** VABC is a regular tetrahedron with edges a. AC and AB are bisected at L and M. What is the area of the triangle VLM ?

* **23.** In the figure ABCD is a regular tetrahedron, each edge of which measures 12 cm. The shaded triangle represents a section through C, E, F. If AE = 4 cm., determine the length of AF that will make the angle FEC a right angle.

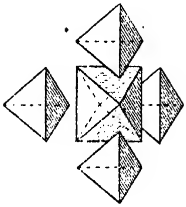
Note.—Remember that the angles EAF, EAC and FAC are each 60°, and express CE^2, EF^2, FC^2 in terms of AC, AE, AF.



* **24.** The mid points of all the six edges of a regular tetrahedron are marked, and a little tetrahedron (with edges half those of the original solid) is cut off each of the four corners. A regular octahedron must be left.

(*a*) What are the comparative volumes of similar solids, in which the dimensions of one are twice those of the other ?

(*b*) What are the comparative volumes of a regular octahedron and regular tetrahedron with the same edge ?



(*c*) Show that the angles between adjacent faces of regular octahedra and regular tetrahedra are supplementary.

[*N.B.*– A Geometrical Proof of this important fact is required in Question **30**. A trigonometrical proof is to be given now.]

(*d*) Calculate the area of an equilateral triangle of side a.

(*e*) Calculate the height of a regular tetrahedron of edge a.

(*f*) How far apart are the opposite faces of a regular octahedron ?

* **25.** [Consult the lower figure on page 468 for the following.]

(*a*) An edge of the cube ABCDPQRS is a ; what is the volume of one of the square pyramids ?

(*b*) What is the volume of the rhombic dodecahedron compared to the cube ?

(*c*) At what angle are the triangular faces of the square pyramids inclined to their square faces ?

(*d*) What are the lengths of a short and of a long diagonal of a rhombus ?

(*e*) What is the size of an acute angle and of an obtuse angle of one of the rhombi ?

(*f*) What is the angle between two adjacent faces ?

(*g*) At one of the corners at which 3 rhombi meet, what is the angle between the short diagonal of one and that of a neighbour ?

(*h*) At one of the corners at which 4 rhombi meet, what is (1) the angle between the long diagonals of adjacent rhombi and (2) the angle between the long diagonals of opposite rhombi ?

(*i*) What is the inclination of the edges DX and CZ ?

* **26.** [Consult the figures on pages 469 and 470 for the following. In calculations about a rhombic dodecahedron go back to the original block, and think out calculations from that.]

(*a*) If a is an edge of the rhombic dodecahedron, find a in terms of s.

(*b*) What are the diagonals of a rhombus in terms of s?

(*c*) What is the area of a rhombus, (1) in terms of s and (2) in terms of a?

(*d*) At how many corners do 4 rhombi meet, and at how many corners do 3 rhombi meet?

(*e*) How far apart (in terms of s) are two opposite faces?

(*f*) How far is each face from the centre of the solid?

(*g*) What is the length of a long diagonal of the solid in terms of s?

(*h*) Give the length of one of its short diagonals in terms of s.

(*i*) Give the long and short diagonals in terms of a.

(*j*) What is the volume of the pyramid, whose base is one of the rhombi and vertex the centre of the solid?

(*k*) What is the volume of the rhombic dodecahedron in terms of s?

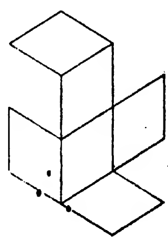
(*l*) How big is the rhombic dodecahedron compared to the original rectangular solid?

(*m*) What is the volume of the rhombic dodecahedron in terms of a?

27. Show that the joins of the centres of the faces of a regular dodecahedron give a regular icosahedron.

28. Show that the joins of the centres of the faces of a regular icosahedron give a regular dodecahedron.

* * **29.** A plane figure consisting of 6 equal rhombi (situated as shown), with angles 60° and 120°, is bent to form a parallelepiped. Show that this is composed of two equal regular tetrahedra (situated at the pointed corners of the parallelepiped) and one regular octahedron (between them).



* * **30.** APQX and APQRSZ are a regular tetrahedron and regular octahedron (with edges of the same length). They have the vertex A and the edge PQ in common, and they are external to each other. Z is the corner, of the octahedron opposite to A. M is the mid point of PQ. Prove that the faces PQX and PQZ are in one plane (in other words, prove that the dihedral angles between adjacent faces of these solids are supplementary).

[HINT. It is easy to show $MX = MA = MZ$, for each is the height of equal equilateral triangles, so that (using the converse of "an angle in a semi-circle is a right angle") it will be sufficient to show $AX^2 + AZ^2 = XZ^2$.]

* * **31.** It is possible to cut down a cube into a regular tetrahedron or into a regular octahedron on one of the following simple plans :

(1) The joins of the ends of opposite diagonals of opposite faces form the edges of a regular tetrahedron (the figure of Ex. **40 b, 9**, on page 434 will give the idea) ;

Or (2) the joins of the mid-points of adjacent faces of a cube form the edges of a regular octahedron (as in Ex. **14** of this Exercise).

Show that both the total area of the surface and the volume of the former are twice those of the latter.

* * **32.** A plumber has only cubical bolts, but of any size. He is required to stop up a pipe of regular hexagonal section with one of these cubical bolts. *He must not cut the bolt at all, nor must he squash it, nor alter its shape in any way.* How can he do the job correctly? If the side of the hexagon is s, what is the edge of the cube?

In particular, if the side of the pipe is $\frac{1}{2}''$, what should be the edge of the bolt? Give the result to the hundredth of an inch.

* * **33.** Calculate the volume of a regular icosahedron, whose edges are a.

CHAPTER XLIV.

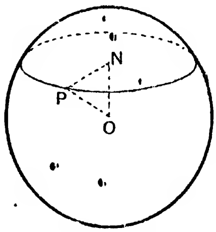
SPHERES.

§ **1.** If points are equally distant from a fixed point they lie on the surface of a **Sphere**. Another conception of the same solid may be obtained by the revolution of a semi-circle about its diameter. A sphere is **not developable.** It is quite impossible to make a piece of paper, without stretching, touch a sphere, all over the surface of the paper *simultaneously.*

§ **2.** That **any plane section of a sphere is a circle** has hitherto been taken for granted. For proof, if O is the centre of the sphere, and if ON is drawn perpendicular to the plane section, and if P is any point on the circumference of the section,

$$OP^2 = ON^2 + NP^2 \text{ (Pyth.)};$$

$$\therefore\ NP^2 = OP^2 - ON^2 = \text{constant}.$$



Now N is a fixed point in the plane.

$\therefore$ the locus of P is a circle having N as centre.

§ **3.** The circle, in which a sphere is cut by a plane, is called a **great circle** when the cutting plane passes through the centre of

the sphere. A sphere is therefore divided into two equal parts by every 'great' circle. Other sections of a sphere give **small circles.**

The enormous importance of 'great circles,' in practice, is fully dealt with in relation to the Earth in Chapter LI. (page 562).

§ **4.** Since plane sections of a sphere are circles, frequently a problem relating to spheres, and in 3 dimensions, is solved by drawing the appropriate section and working in 2 dimensions. Sometimes, by working first in one section, and then in another, we are able to get properties of a sphere. Proposition **45** is an instance.

In the same way that we have *tangent lines* to a circle we have *tangent planes* to a sphere, and **various properties of a circle have their counterparts in the properties of a sphere.**

In the case of circles, equal chords are equidistant from the centre; for spheres equal plane sections have the same property.

For circles, equal arcs are cut off by equal chords and equal areas of spheres are cut off by equal plane sections.

A *tangent line* touches a circle at one point, and at one point only, and is at right angles to the radius at the point of contact; in the same way a *tangent plane* touches a sphere at one point, and at one point only, and is at right angles to the radius at the point of contact.

The line joining the centres of two circles touching passes through the point of contact; the same property belongs to two spheres touching.

And so it comes to this:

Chord properties of a circle correspond to the properties of planes cutting spheres.

Tangent properties of a circle correspond to the properties of planes touching spheres. And, above, we have seen that two circles (in plane Geometry) touching has its counterpart in the contact of two spheres (in solid Geometry).

In addition, we have the case of lines touching spheres; but, as plane sections of a sphere are circular, we apply 2-dimensional properties to these direct.

It should be noted that, whereas two (necessarily equal)

tangent lines can be drawn to a circle from a given external point (and in the plane of the circle), an infinite number of straight lines can be drawn, from a given external point, to touch a sphere. The lengths of straight lines (from the external point to the points of contact) are equal. The straight lines form a **tangent cone** to the sphere.

But the *Angle* properties of circles have no direct counterpart for spheres.

PROPOSITION 45.

§ 5. **General Enunciation.** *If chords of a sphere intersect at a given point (inside or outside the sphere) the rectangle contained by their segments is constant in area; and, in the case where the point of intersection is outside the sphere, the square on the tangent (from that point to the sphere) has the same area.*

Particular Enunciation. AB, PQ and XY are three of the chords cutting at K. And, when K is outside the sphere, KT is one of the tangents.

To prove rect. KA . KB = rect. KP . KQ = rect. KX . KY = KT^2.

[No figure conveys the idea so well as a model, and the imagination of the model is generally sufficient.]

Proof. Any *pair* of chords intersect at K, so that a plane can be imagined through that *pair*.

Plane sections of spheres are circles, so that we can apply the properties of circles to any *pair* of these lines (for they intersect).

Now, if the plane containing AB and PQ, and also the plane containing PQ and XY, be considered in succe

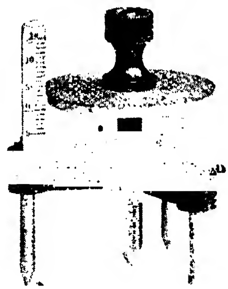
rect. KA . KB = rect. KP . KQ = rect. KX

In particular, if a secant from K cuts the sphere at two points coincident at T, rect. KX . KY = KT^2

These arguments establish the truth of the theorem.

Q.E.D.

* § 6. In practice, when it is possible, we are apt to measure *diameters*, as the centre (for radii) is often not well marked and is frequently un-get-at-able [*e.g.* footballs]. Sometimes only pieces of spheres are available [*e.g.* lenses], so that then diameters are not possible. From suitable measurements it is easy to calculate the radii. A figure of a **spherometer** is given.



The 4 legs, by screwing, are made to touch a sphere simultaneously and the difference of the level of the centre leg (movable) and the 3 outside legs (fixed) is read; whence we obtain data for calculating the unknown radius.

* § 7. First screw the centre leg to flatness with the other legs and get the "zero-reading." And also let the distance from the centre to the outside legs be s.

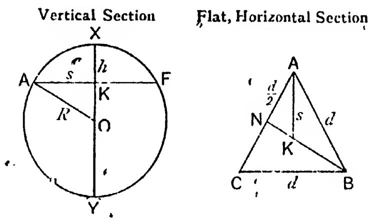
Secondly, consider the following figures:

The four legs of the instrument are made to touch the surface of the sphere simultaneously. The points of contact are A, B, C and X.

In the vertical section, rect. XK . KY = rect. AK . KF ;

$$\therefore\ h(2R-h)=s^2,$$

$$\text{whence}\quad R=\frac{h}{2}+\frac{s^2}{2h}.$$



In the horizontal section, ABC is an equilateral triangle, so that KN is half KA ;

$$\therefore\ s^2=\left(\frac{s}{2}\right)^2+\left(\frac{d}{2}\right)^2 \text{ (Pyth., } \triangle\text{AKN)};$$

$$\text{whence } s^2=\frac{d^2}{3};$$

$$\text{so, by substitution,}\quad R=\frac{h}{2}+\frac{d^2}{6h}.$$

[Now this d is always the same for the same spherometer, and it is as well to calculate $\frac{d^2}{6}$ and $\log\frac{d^2}{6}$, once and for all, and write them on the spherometer box.]

* § **8.** An **Example** of calculation is appended.

In a spherometer $d = 4$ cm. The "zero-reading" was $-0{\cdot}02$ mm. (average of several observations).

When all 4 legs were in contact with a convex spherical lens, the (average) reading was $+3{\cdot}74$ mm. What is the radius of curvature of the lens?

Reading of the centre leg $= +3{\cdot}74$ mm.
Zero reading $= -0{\cdot}02$ mm.

$\therefore\ h = 3{\cdot}76$ mm.

[Notice that you subtract the zero reading.]

$$\therefore\ h = 0{\cdot}376 \text{ cm.}$$

$$\text{and}\quad d = 4 \text{ cm.}$$

$$\text{Now}\quad R = \frac{h}{2} + \frac{d^2}{6h};$$

$$\therefore\ R = \frac{0{\cdot}376}{2} + \frac{16}{6 \times 0{\cdot}376}$$

$$\text{whence}\quad R = 7{\cdot}28\ldots.$$

Hence the radius of curvature is about 7·28 cm.

[Of course, for choice, use tables of logarithms to shorten the mere computation.]

You must be especially careful about the zero reading for concave surfaces.

* § **9. The necessity for a very accurate value of h is the reason for the nicety of the instrument.** Whole numbers are read on the vertical scale. The screw has a very small pitch, and the circular table is enlarged so that hundredths can be read easily.

The spherometer is an elegant instrument, which substitutes the sense of touch for that of sight.

There are, naturally, instruments of various patterns. In any particular spherometer it may well happen that not one turn (perhaps two) means 1 unit on the vertical scale, hence you should find out all about a particular instrument before you trust it too implicitly.

EXAMPLES 44 a (CALCULATIONS. SPECIAL CASES).

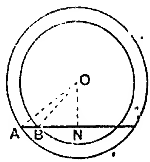
1. A rubber ball, of diameter $2\frac{1}{2}''$, is squashed between two (parallel) window panes, which are $2''$ apart. The squashing, on each side, is the same. What are the diameters of the circles of contact with the panes? [In the calculations be sure to avoid mixing up diameters and radii.]

2. From a given point K, outside a given sphere, secants KAB, KPQ and KXY are drawn, in various directions, cutting the sphere, and also the line KT touching it.

It is given that $KA = 4''$, $AB = 140''$, $KP = 8''$, $KY = 48''$. How long are KQ, KT and XY?

[Do not mix up AB and KB, etc.]

3. A spherical cheese is 10 cm. in *diameter* and has a rind 1 cm. thick.



A slice, of maximum thickness (including the rind) 2 cm., is cut off.

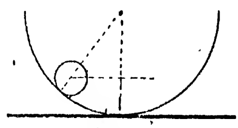
What is the width of the exposed rim?

[HINT. $AB = AN - BN$. Work with the triangles OAN and OBN.]

4. Water, to a maximum depth of 2 inches, is in a hemispherical basin of *diameter* 13 inches. How long is the water line, that is the circumference of the surface of the water?

5. A hemispherical bowl, of radius 25 cm., is fixed.

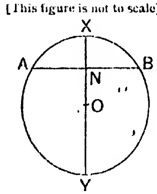
A spherical ball, of radius 4 cm., is revolving inside the bowl, at such a velocity that its centre is always 9 cm. above the horizontal plane through the lowest point of the bowl. What is the length of the locus of the centre of the ball, for one of its revolutions round the surface of the bowl?



How far is the point of contact from the central axis?

6. A golf ball has diameter 1·62″. Its name is on its surface n a circle.

[This figure is not to scale]



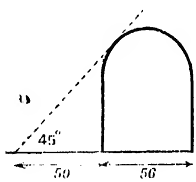
AB is a diameter of that circle, whose entre is N. XA = $\frac{1}{4}$″.

When the ball is struck, there is impressed n the face of the club the negative of the ircle with the name of the ball.

To what extent is the ball squashed in by he impact ? Give the result to the nearest undredth of an inch.

[It is easiest, in this case, to work with XA ather than NA, but these are sensibly the same in practice ; also IY and XY are practically the same.]

7. A cylindrical building, of diameter 56 feet, is surmounted by a hemispherical lome. At 50 feet away from the wall of he building, the elevation of the *apparent* op is 45°. How high is the building entrally ?



8. Two spheres, with radii 17″ and 7″, have their centres 5″ apart. Calculate the radius of a sphere which touches both, and which has its centre on the line of centres of the given spheres. There are 4 cases to consider.]

9. A sphere, with diameter 5″, rests on a triangular hole in a board. The sides of the triangle are $8\frac{1}{2}$″, 8″ and $7\frac{1}{2}$″. What is the maximum height of the sphere above the board ?

[Hint. You need to calculate first the radius of the circle nscribed in the triangle.]

* **10.** A spherometer (in which the 3 outer legs form an equilateral triangle with sides 4 cm.) is used to find the radius of curvature of a convex lens. The " zero reading " was +0·34 mm. In contact with the lens, the reading is +3·54 mm. What is the radius of curvature of the lens ?

* **11.** With the same spherometer as in the preceding question, and with the same zero reading, the radius of curvature of a portion of a wash-hand basin (inside) is to be found. In contact with the basin the reading is −2·84 mm., what is the radius of curvature of that portion ?

* 12. In the preceding question, if the "zero reading" had not been taken into account, what radius of curvature would have been deduced ?

* 13. An ordinary drinking glass, 5 inches in internal vertical height, has internal diameters of 3 inches at the top and 2·5 inches at the bottom. The inside of the bottom is flat. A sphere has to be made which, when dropped into the glass, will rest on the bottom and just touch the sides of the glass. Find the diameter of the sphere, and the height of the circle of contact between the sphere and the glass above the bottom of the glass.

* 14. Two spheres have radii 13 cm. and 9 cm. Their centres are 8 cm. apart. What is the radius of a sphere which touches both internally ? [Two cases.] The three centres are in the same straight line.

* 15. With spheres of the size and position of those in the previous question, what is the radius of a sphere which touches the larger internally and the smaller externally ? It is to have its centre in the line of centres of the given spheres.

* 16. The circumference of an Association football being $27\frac{1}{2}$ inches, find how far a fly should be from it to make $\frac{1}{4}$ of its surface visible.

* * 17. A cubical box has edges 1 foot long. Into it it is possible to pack little spheres, each of diameter 1 inch, in either of the two following ways:

(1) In one system there are 12 layers, of 144 in each layer, each sphere being vertically above the one below.

(2) In the other system there are 144 in the lowest layer, 121 in the layer above (each sphere sinking into the interstice formed by four spheres in the layer below). In the third layer there are 144. In the fourth layer 121, and so on alternately, until there is no room for another layer.

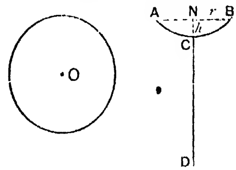
How many spheres can be packed under each of the two systems ?

[Notice that there are getting on for a quarter more under the second system than under the former.]

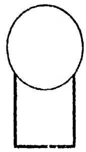
* * 18. A croquet ball (diameter $3\frac{5}{8}$ inches) rests on a triangular hole in a board. The sides of the triangle are 10, 9 and 4 inches. What is the maximum height of the ball above the board ? Give the result correct to an eighth of an inch.

EXAMPLES 44 b (CALCULATIONS. GENERAL CASES).

1. ACB is a cup for "cup-and-ball." Into the cup the ball O fits tightly. If AB $=2r$ and NC $=h$, where NC is the maximum depth of the cup (which is part of a sphere), prove that the radius of the ball O is given by $\dfrac{h^2+r^2}{2h}$.



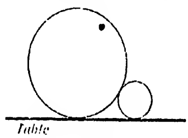
2. A spherical ball (diameter $2R$) rests on the mouth of a vertical cylindrical glass (diameter $2r$). $R>r$. To what depth does the ball penetrate into the glass?



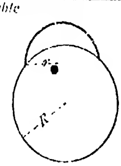
3. A sphere, of radius R, rests on a circular hole, of radius r; how far is the top of the sphere from the plane of the circular hole?

4. What are the radii of the circum-sphere and in-sphere of a cube whose edges are a?

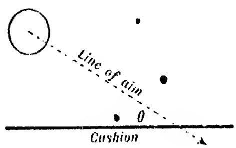
5. Two spheres, of radii R and r, are lying on a flat table touching each other. The larger is fixed, and the smaller circles round it touching it and the table all the time. Find the length of the locus of its centre for one revolution.

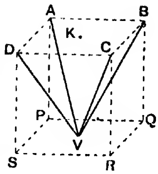


6. A hollow hemisphere of radius r is put, like a cap, on to a sphere of radius R; but (r being smaller than R) the cap does not fit. What is the maximum distance between the surfaces?



7. If r is the radius of a billiard ball, show that the point of contact with the cushion must be $r\cot\theta$ behind the intersection of the line of aim and the cushion. In particular, if the diameter of the ball is $2\frac{1}{16}$ inches, and the angle is $30°$, show that the point of contact is over $1\frac{3}{4}$ inches behind the point aimed at on the cushion. [This is assuming the height of the contact with the cushion is a half ball.]



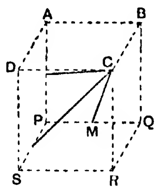


*** 8.** Stiff wires are stretched from the centre of each face of a cube to the 4 corners of the opposite face (only 4, of the total 24, namely VA, VB, VC and VD, are shown in the figure). What is the radius of the largest sphere that can be put in to touch all the wires ? The edge of the cube is a.

[HINT. Work with the plane SDKBQV. Inscribe a *circle* in the rhombus contained by the wires VD, VB, KQ and KS.]

*** 9.** In the preceding example, if BVC, CVD, DVA and AVB are stiff plane faces, what is the radius of the largest hemisphere (whose flat top is in the plane ABCD) that can be got into the funnel formed by these plane faces ?

[HINT. If M and L are the mid points of the edges DA and BC, work with the plane MKLV.]



*** 10.** From each of the corners of a cube (whose edge is a) issue 3 stiff wires, stretching respectively to the mid points of the 3 edges, which meet at the diagonally opposite corner of the cube. [Only the 3 wires issuing from *one* of the 8 corners are shown in the figure.] What is the radius of the largest sphere that touches all the 24 wires ?

[HINT. If O is the centre of the cube, you need to find the perpendicular from O to the line CM. The plane CPMQ contains O.]

*** 11.** A pair of compasses is open to draw a (plane) circle of radius s. It is used, at that opening, to describe a circle on the surface of a sphere of radius R. What is the radius of the circle drawn ?

[*N.B.*—Of course the answer should come to s exactly, when R is put equal to infinity, for that would be drawing a circle on a plane. You should test your answer thus.]

*** 12.** Two pyramids are formed of tennis balls. First 3 are put in a horizontal plane, touching each other, in the shape of an equilateral triangle, and a fourth is laid on the top. Secondly,

4 are put in a horizontal plane, touching each other, in the shape of a square, and a fifth is laid on the top.

Which is the taller of the two pyramids? If the diameter of the tennis ball is d, what are the two heights?

***13.** A packing case measures $8d$ by $5d$ by $6d$ (deep), where d is the diameter of a ball. 240 (*i.e.* $8 \times 5 \times 6$) balls are packed in regularly, in 6 layers of 40. Another plan is adopted as follows: 40 (*i.e.* 8×5) are packed into the lowest layer, then above 28 (*i.e.* 7×4), each ball sinking into the hollow made by the four balls below it. In the third layer are 40, in the fourth layer 28, and so on alternately. How many layers can there be in the box? What number of balls can be packed if this plan is adopted?

[HINT. Notice that the centres of four balls, touching and forming a square in any layer, and the centre of the ball immediately over them, in the layer above, form the angular points of half a regular octahedron. It is the height of this pyramid that will give the distance between layers.]

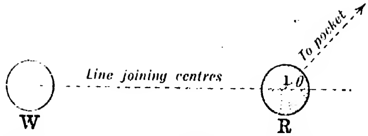
***14.** The smaller (radius r) of two spheres is entirely within the larger (radius R), and their centres are c apart. What is the radius of the largest and smallest spheres which touch both?

***15.** A regular octahedron has edges a. What is the radius of its circum-sphere? What is the radius of its in-sphere?

***16.** What is the radius of the greatest sphere that can be cut from a cone whose height is h and base-radius r?

***17.** The centres of two cutting spheres, with radii R and r, are c apart. Calculate the radius of a sphere which touches both internally, and which has its centre on the line of centres of the given spheres. [There are two cases to consider.] Now, if the contact were external in one case and internal in the other, what would be the two answers?

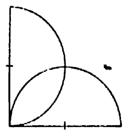
***18.** Two spheres, with radii R and r and with their centres c apart, are entirely outside each other. Calculate the radius of a sphere which touches both and which has its centre on the line of centres of the given spheres. Distinguish between external and internal contact. [There are 4 cases to consider.]



*19. To make it possible for **W** to force **R** into the pocket direct, show that $\theta < \cos^{-1}\dfrac{d}{s}$, where d is the diameter of the billiard balls **W** and **R**, and s is the distance between their centres.

In particular if $\theta = 60°$, show that the centre of **W** must be over two diameters from the centre of **R** to make the direct "winning hazard" possible (*i.e.* for **W** to hole **R** direct). It is assumed that the balls run in straight lines when struck.

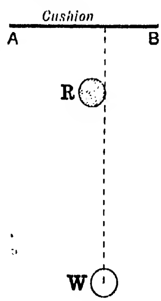
*20. Two equal hemispheres (with radii r) have their plane faces at right angles and are situated, as in the figure, cutting each other. Their centres are in a plane at right angles to both their plane surfaces. A plane touches both hemispheres. How far apart are the points of contact?



*21. A cubical box just contains a sphere. The sphere is cut into octants, one of each of which is fixed into a corner of the cube, so that the curved portions are all inwards. Now another sphere is put inside, as great as possible, and hence touching each of the eight octants. If an edge of the cube is a, what is the diameter of the second sphere?

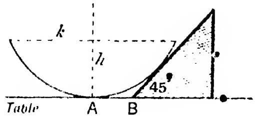
[If three planes (mutually at right-angles) pass through the centre of a sphere, they divide the solid into 8 portions called Octants.]

22. The cue ball **W is struck, quite centrally, and makes a "half ball-shot" at **R**; the direction of the original course of **W** (besides being tangential to **R**) is exactly perpendicular to the cushion AB. **R** is struck hard enough to reach AB and rebound from it. Assuming perfect elasticity between **R** and the cushion and perfect smoothness of cloth, draw a diagram showing the course of **R**. [It is to be assumed, too, that the balls are mere points in size.]



**** 23.** A piece of toast, with one angle 45°, is held perpendicular to a table, and is pushed as far as it will go, without squashing it, under a saucer (which is a segment of a sphere).

k and h (see the figure) define the size of the saucer. What is the length of AB ?



[You will notice that saucers of this shape will naturally tilt the more the toast is pushed, yet with saucers of such a shape the distance AB is the same whatever be the tilt.]

**** 24.** A hemispherical bowl has radius $4r$. In it, as low as possible, are three equal spheres, of radius r, touching each other. On the top of the equilateral triangle of the three spheres, thus formed, is placed another equal sphere to form a pyramid. Is the top of this pyramid above or below the centre of the hemisphere? In either case state the difference, in terms of r, correct to three significant figures.

**** 25.** Three equal spheres (of radii r) are arranged, touching each other, as an equilateral triangle, and a fourth equal sphere is placed on them as a pyramid. What is the edge of the regular tetrahedron which would just enclose them ?

[HINT. Recollect that the centres of the four spheres form one regular tetrahedron, and consider how much this should be enlarged.]

**** 26.** A point of light is situated at a distance l from the centre of a sphere whose radius is r. The rays of light form a tangent cone to the sphere. Show that the bright area of the surface of the sphere is $2\pi r^2\left(1-\dfrac{r}{l}\right)$. [In particular, if the distance of the light is very great compared to the radius of the sphere, show (from this) that practically a hemisphere is illuminated.]

**** 27.** A cubical box just contains a sphere of diameter $2R$. Into the 8 corners of the box are *then* fitted spheres of the largest size that will go in. What is the radius of that size ?

**** 28.** Find the radius of a sphere inscribed in a given octant of a sphere whose radius is R.

**29. ABCD is a square with sides r. Two equal spheres, with radii r and centres A and C, are described. Plane sections are made of the solid, either (1) by the plane ABCD or (2) by a plane (through B) perpendicular to the line AC. What is the ratio of the two areas?

**30. Two equal spheres (with radii R and their centres also R apart) have coalesced. A and B are points, common to both spheres, and at opposite ends of a diameter of their plane section. A plane section is also made through A, perpendicular to the line AB. Sketch the perimeter and find its length.

**31. VABC is a regular tetrahedron with edges a. Find the radius of the sphere which *circumscribes* it.

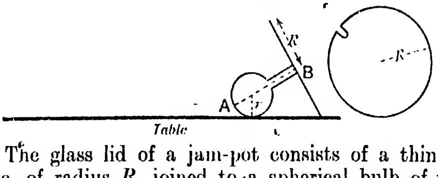
[HINT. Circumscribe the base equilateral triangle ABC with a circle. Let its centre be S. Let AS, produced cut BC at M and the circle again at X. This circle is one of the "small" circles of the sphere. Again draw another section VAX. Let O be the centre of the circle circumscribing the triangle VAX. That circle is a "great" circle of the sphere.]

**32. VABC is a regular tetrahedron with edges a. Find the radius of the sphere *inscribed* in it.

[HINT. Read carefully the hint to the previous question. It is the perpendicular from O to the line VM that you need.]

**33. What is the radius of a sphere which touches a plane containing an equilateral triangle ABC, with sides R, and each of three spheres (with radii R), and with points A, B and C as centres?

[*N.B.*—Draw the plane section ABC. You will notice that there are 3 cases to consider.]



**34. The glass lid of a jam-pot consists of a thin circular disc, of radius R, joined to a spherical bulb of radius r. AB $= k$.

The lid is resting on a table, as shown. In the edge of the lid there is a U-shaped hole for the spoon. The lid

makes one complete revolution on the table; how many revolutions of the lid does the U make?

[HINT. Produce BA to meet the table at C. Let AC be equal to b and be inclined to the table at an angle θ. Eliminate θ and b.

EXAMPLES 44 c (RIDERS).

O **1.** What is the locus of points in a plane which are equidistant from a given point outside the plane?

O **2.** What is the locus of a point in space the sum of the squares of whose distances from two given points is constant?

O **3.** How many points are there equidistant from four given points which are not in one plane?

O **4.** What is the locus of the feet of perpendiculars, from a given point, to planes which pass through a given point?

O **5.** What is the locus of the centres of all spheres which touch a given plane at a given point?

O **6.** Show how to describe a sphere, of given radius, touching a given sphere and two given planes. Discuss the limits of possibility. What is the maximum number of solutions?

7. Prove that the section of two intersecting spheres is a circle whose plane is at right angles to the line of centres of the spheres.

8. O is the centre of a sphere. OM and ON are drawn perpendicular to two equal plane sections of the same. Prove that the triangle OMN is isosceles.

9. AB is a diameter of a given sphere. AM and BN are drawn perpendicular to *any* plane section of the same. The straight line MN meets the sphere at the points P and Q. Prove that MP = NQ.

10. If two spheres are concentric, any tangent plane of the inner will cut the outer in a circle of constant radius.

11. A description of the figure follows:

R = object ball (at billiards) and its centre.

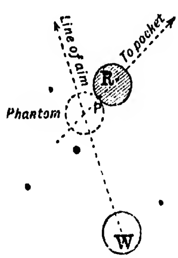
W = cue ball and its centre.

P = point of contact.

(The phantom ball, into the position of which **W** must come, is shown dotted.)

The object is to make a "winning hazard," *i.e.* for **W** to force **R** to run into the pocket.

Show that the perpendicular, from the centre **R** to the line of aim, is exactly double the perpendicular from P to the same line.



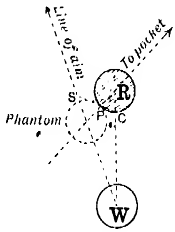
12. A and B are the centres of two non-intersecting equal spheres. The straight line AB is bisected at M. Any plane passes through M and cuts the spheres. Prove that the areas of the plane sections of the spheres must be equal.

13. KAB is a straight line cutting a given sphere at the points A and B. KT is a tangent line to that sphere. Prove that KT is a Mean Proportional between KA and KB.

14. Two spheres have the same centre. Show that tangent lines drawn from any point on the outer to the inner are equal in length.

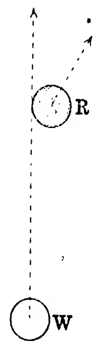
15. The description of the figure in Question **11** is the same for this, and in addition C (on the *surface* of **R**) is the apparent centre of **R** as seen from **W**. CP produced cuts the line of aim at S. Prove that $PS > PC$.

[As a matter of fact PS and PC are practically equal if **W** is (say) 10 diameters or more from **R**.]



16. Two spheres touch externally at K; and two straight lines XKY and PKQ, pass through their points of contact and cut one sphere at X and P, and the other at Y and Q. Prove that the lines XP and YQ are parallel.

17. **W** and **R** are two billiard balls on a billiard table. **R** is at rest. **W** is struck (quite centrally) along a tangent to **R** making a "half-ball-shot." Show that **R** (when struck by **W**) must move in a straight line, which makes with the original path of **W** (*prolonged*) an angle of 30°.



18. K is a point external to a sphere whose centre is O. TS is a diameter of the base of the tangent cone (whose vertex is K). TS and KO cut at N. Prove that TN is a fourth proportional to KO, KT and TO.

* 19. Tangent lines are drawn to a sphere from a given external point. Prove that (1) the tangents are equal in length, and (2) that the points of contact lie in a circle.

* 20. P is a variable point on a fixed plane; O is a fixed point not in that plane. Q is a point in OP such that the rectangle OQ . OP is constant. Prove that the locus of Q is a sphere.

* 21. Find the locus of points in a given plane at which a straight line of fixed length and position subtends a right angle. Suppose that the straight line is entirely outside the plane.

* 22. The six planes bisecting the edges of any tetrahedron at right angles will meet in a point. What is that point called?

* 23. How should a plane be drawn through a given point, within a given sphere, so that the area of the section of the plane by the sphere should be a minimum?

* 24. Given two points on the surface of a given sphere, show how to describe the great circle passing through them.

* 25. If three spheres cut one another, prove that their planes of intersection meet in a straight line perpendicular to the plane determined by their centres.

Prove that tangent lines, to the three spheres, from any point on this line must be equal in length.

* 26. V, A, B and C are the angular points of a tetrahedron, two of whose edges, namely VA and VB, are equal. VA is bisected at M. A sphere passes through B and touches the edge VA at M. The sphere cuts the edge VB at K. Prove that VK = ¼VB.

* 27. Two segments of spheres are on the same (circular) base and on the same side of it. A series of parallel straight lines passes through points on the boundary of that circular base; one of these lines cuts the two segments at the points B and C. Prove that the length BC is constant.

[Hint. Consider the parallel straight lines to be vertical, and prove that BC is twice the difference of the levels of the centres of the two spheres.]

* 28. Three chords of a sphere, whose centre is O, mutually at right angles, meet at K. Prove that twice the sum of the squares on the 6 segments of the chords is equal to the sum of the squares on those 3 chords together with 4 times the square on KO.

** **29.** The joins of the mid-points of opposite edges of *any* tetrahedron are the diagonals of an octahedron, (see page 427). These diagonals divide the octahedron into 8 tetrahedra, whose bases are the faces of the octahedron and whose vertices are coincident at the intersection of the joins. Prove that the centres of the circum-spheres of the tetrahedra must be the corners of a parallelepiped.

** **30.** A corner is cut off a cube, the section being a triangle. Prove that the perpendicular from this corner of the cube, to the plane of the section, meets it at the orthocentre of the triangle.

[HINT. ABC is the triangle and O the point at the corner of the cube. $\angle$BOC = rt. $\angle$; $\therefore$ pt. O is on the sphere whose diameter is BC. Likewise O is on the sphere whose diameter is CA. These two spheres cut in a plane at right angles to the line joining their centres, the mid points of BC and CA, and this line is parallel to AB; so that this plane is perpendicular to AB, etc.]

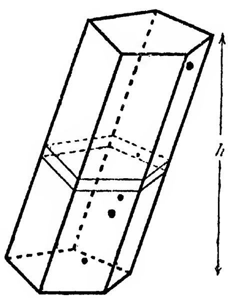
** **31.** Three straight lines are mutually at right angles, and issue from a given point. They cut a given sphere. Prove that the sum of the squares on the three chords is constant.

** **32.** Three straight lines VA, VB and VC are mutually perpendicular. A, B, and C are fixed points on them. VK, VL and VM are perpendicular to the lines BC, CA and AB respectively. Prove that AK bisects the angle LKM.

CHAPTER XLV.

AREAS AND VOLUMES (PRISMS AND PYRAMIDS).

§ 1. The **Areas** of the surface of solids, bounded by plane faces, afford no difficulty. Our knowledge of Plane Geometry gives us a means of determining the area of each face separately, and **totalling** up gives the required result. If several faces are the same, clearly the arithmetic is lessened.

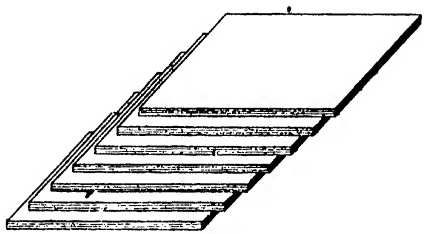


The area of the curved surface of a cylinder, which is developable, can be imagined easily by wrapping a piece of paper round it. On unwrapping we have a rectangle, and thus

Area of the curved surface of a Cylinder $= 2\pi rh$.

§ 2. That the **Volume of a Prism** is equal to the area of its base multiplied by its height (*measured perpendicular to the base*), whether the prism is oblique or not, is easily seen by dividing it into *thin* plates parallel to the base, as is indicated in the figure on page 497, finding the volume of each plate and totalling up. Now, since the area of each plate is the same as that of the base,

Volume of Prism = base × height.



The plates have a stepped appearance, if they are appreciably thick; but, if they are absolutely thin, this appearance is lost. It is easy with a pile of books of the same size (or, better still, with a pile of single sheets of paper) to convince ourselves that the volume is unaltered by sloping the pile.

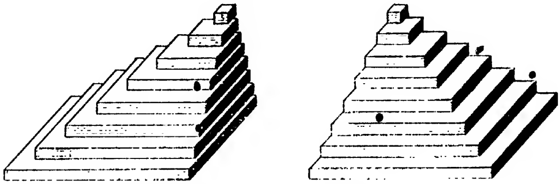
It should be noted that a Prism may have a polygon of any shape for base. Many (but by no means all) glass prisms have triangular sections, and one is inclined to apply the word prism only to this special kind. This is wrong.

The **Cylinder** is only a special case (but a very important one). The base is circular, so that the

Volume of a Cylinder $= \pi r^2 h$.

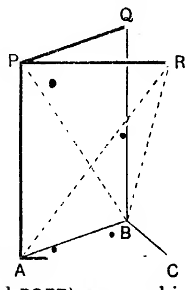
[Only the case of *circular* bases for cylinders is considered in this book.]

§ 3. The Volume of a Pyramid is $\frac{1}{3}$ the Volume of the Prism on the same base and of the same height. This very important fact is most shortly *proved* using the notation of the Integral Calculus, and that proof is given in full in § 4.



In default of that consider the following:

The pyramid is imagined as composed of slabs parallel to its base. The slabs, simply pushed across a little, will make a pyramid with the same height and with the same base. Clearly the volume is unchanged (consisting, as it does, of the same slabs). Hence one argues that pyramids of the same vertical height, and on equal bases, must have the same volume. [The "stepped" appearance is lost if the slabs are infinitely thin.] *Now, whatever the shape of the base of the pyramid, it can be reckoned (always) as a cluster of pyramids on triangular bases,* and so it is only necessary to consider the relation between a right prism on a triangular base and the pyramid on the same.



Vol. of the one Prism PQRABC = Vols. of 3 pyramids ABCR, PABR and PQRB.

Plainly the two pyramids ABCR and PQRB are equal in volume, for they are on congruent bases ABC and PQR and of the same altitude.

Also, the two pyramids PABR and PQRB are equal in volume, for they are on congruent bases PAB and BQP and of the same altitude.

Hence the *three pyramids* (ABCR, PABR and PQRB) are equal in volume, and together they make the *one prism* (PQRABC).

$\therefore$ **Volume of Pyramid = $\frac{1}{3}$ base area × height.***

* This truth was established by Eudoxus, who flourished about the middle of the 4th century B.C.

It should be noted that a pyramid may have a polygon of any shape for its base. The Pyramids of Egypt have square bases, and they are such familiar examples that one is inclined, at first, to believe that pyramids must have square bases. This is not the case.

If the base of the pyramid is a circle, the solid is called a **Cone**, so that the

$$\textbf{Volume of a Cone} = \tfrac{1}{3}\pi r^2 h.$$

[Only the case of *circular* bases for cones is considered in this book.]

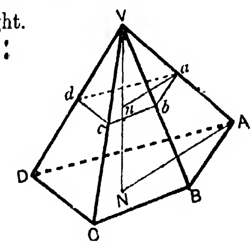
The following section should be considered as an alternative to the preceding.

*** * § 4. A formal proof, that the volume of a pyramid is $\frac{1}{3}$ that of the corresponding prism, is now given.**

$abcd$ is a section parallel to the base ABCD.

VnN is the perpendicular height.

V$n = x$, VN $= h$.



As the section $abcd$ and the base ABCD are similar, their areas are proportional to the squares of corresponding lines, so that

$$\text{the area of } abcd \propto (an)^2,$$
$$\propto (\mathrm{V}n)^2 \text{ for } an : \mathrm{AN} = \mathrm{V}n : \mathrm{VN},$$

i.e. area of $abcd = kx^2$, where k is some constant (and the area of the base ABCD $= kh^2$).

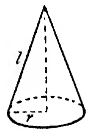
Now imagine the pyramid built up of flat plates, parallel to $abcd$, and that the thickness of the $abcd$ plate is δx;

$$\therefore \text{ Volume of the Pyramid VABCD} = \int_0^h kx^2 dx$$
$$= \left[\tfrac{1}{3}kx^3\right]_0^h = \tfrac{1}{3}kh^3 = kh^2 \times \tfrac{1}{3}h;$$

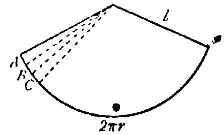
$$\therefore \textbf{ Volume of a Pyramid} = \textbf{Base} \times \tfrac{1}{3}\textbf{ height.}$$

§ **5.** Since a *cone is developable* it is easy to find the area of its curved surface.

Cone.



Paper (which just covers the curved surface) flattened out.



By flattening out the paper a fan-like sector is obtained. That sector can be imagined as a multitude of triangles with bases A, B, C, etc.;

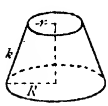
$\therefore$ the area of the sector $= \frac{1}{2}(A + B + C + \ldots)\, l = \frac{1}{2} \times 2\pi rl.$

Hence the

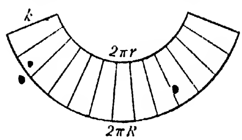
Area of the curved surface of a Cone $= \pi$rl.

§ **6.** For a frustum of a cone we can imagine the curved surface

Frustum of a cone.



Paper (which just covers the curved surface) flattened out.

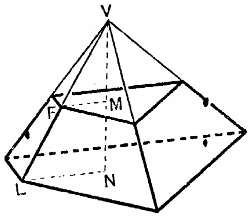


unwrapped, for *the frustum is developable*, and laid flat and then cut into a multitude of trapezia, from which

Area of curved surface of a frustum of a Cone
$=\frac{1}{2}$ **sum of circumferences** $\times$ **slant height.**

[The word " frustum " means " a bit." The spelling frustrum was not uncommon, but erroneous. In Geometry the word is used when the vertex of a pyramid, or cone, is cut off by a plane parallel to the base.]

§ 7. For the **Volume of a frustum** of a pyramid (including a cone), imagine the solid completed, and calculate its height by similar triangles. After that subtraction gives the volume of the frustum.



For instance, in the figure, the triangles VMF and VNL are similar;

$$\therefore \frac{VM}{VN} = \frac{FM}{LN};$$

$$\therefore \frac{VM}{VN - VM} = \frac{FM}{LN - FM};$$

$$\therefore \frac{VM}{MN} = \frac{FM}{LN - FM},$$

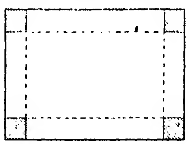
so that VM can be found, and then, etc.

(*N.B.*—*Any* corresponding lines can be used for the second ratio; for cones they are often radii or diameters.)

EXAMPLES 45 a (CALCULATIONS. SPECIAL CASES).

Prisms.

1. The figure represents a rectangular sheet of zinc 36 cm. long and 24 cm. wide.



From each of its corners a square piece is cut, and the edges are then bent up at right angles, along the dotted lines, to form an open tray. The joints are soldered so that it will contain water without leaking.

If the length of the side of each square piece cut off is x cm., give the length, breadth and depth of the tray.

Find an expression (containing x) for the volume of the tray in cubic cm. (Leave this expression in factors and call it V cu. cm.)

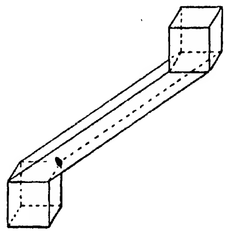
Make a table of values of each of these factors and of V.

Illustrate graphically how V varies with x.

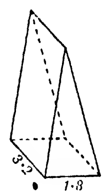
From the graph determine the depth of the tray (or length of the side of each square cut away) if it contains exactly 1 litre, *i.e.* 1000 cu. cm.

From the graph determine what should be the depth in order that the tray may have a maximum capacity. What is this maximum capacity?

2. A piece of metal is of the shape indicated. The two ends are cubes whose edges are 3 cm. The furthest faces of the cubes are perpendicularly 13 cm. apart. What is the volume of metal?

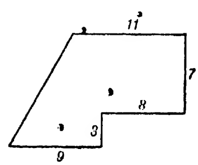


3. An iron wedge has a base 1·8″ by 3·2″, and ends isosceles triangles, whose bases are 1·8″ and vertical height 4″. Calculate the volume and total surface of the wedge.

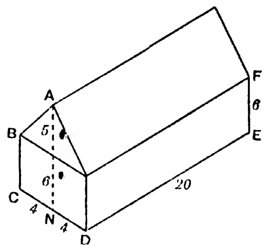


4. A bottle consists of a cylinder (radius 2 cm. and height 3 cm.) with a cylindrical neck (radius 0·5 cm. and height 1·5 cm.) above it. Calculate its volume (to the nearest cu. cm.) when it is filled to overflowing.

5. A pile of paper is made of pieces of the shape shown. The dimensions are in inches. The pile is on a horizontal table, and its edges slope at an angle of 30° to the vertical. The length of its slant edges is 4″. Give the volume of the pile to the nearest cubic inch.



6. The wooden shed of the accompanying pattern and without a floor is 20 ft. long, 8 ft. wide, and its height varies from 6 ft. to a maximum of 11 ft. at the centre.



(*a*) Calculate the length of AB.

(*b*) Calculate the number of square feet of wood needed, and add an extra 10 per cent. to this to allow for waste in cutting, etc.

(*c*) If the shed is made of match-boarding 6 inches wide, what length must be ordered to build such a shed?

(*d*) Draw to scale a plan of the shed and measure NE.

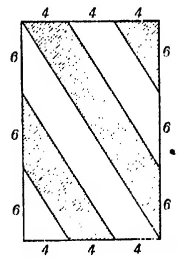
(*e*) Calculate NE to the nearest tenth of a foot.

(*f*) Draw a vertical section through ANEF and measure AE.

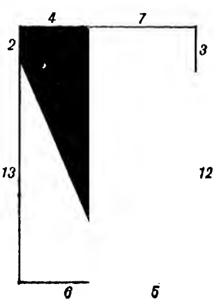
(*g*) Calculate AE to the nearest tenth of a foot.

(*h*) Find the volume of the shed in cubic feet.

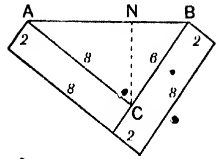
7. Sections of three beams (all 8 ft. long) are shown shaded. The dimensions are in inches. Give the volumes of each beam in cubic feet.



8. A wooden rectangular panel, 15″ by 11″, has the cross-section indicated. The dimensions are in inches. The wood is $\frac{1}{4}$″ thick. If the weight of the wood is 48 lb. per cu. ft., find the weight of the black portion of the panel. Give the result to the nearest ounce. [The black portion is cut from the panel.]



9. A board 6 ft. 6 in. long, 8 in. wide and 2 in. thick is sawn into three pieces 6 in., 3 ft. and 3 ft. long respectively. The smallest piece is cut along one of its diagonals into two triangular pieces. The four pieces are then nailed together to form a trough.



The diagram shows a section of the end of the trough—dimensions in inches.

(*a*) Draw, to scale, a figure showing the board before it is cut, and on it mark the lines through which it has to be sawn.

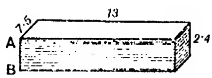
(*b*) Calculate the vertical height CN.

(*c*) If CN is divided into 4 equal parts, calculate the volume of water (in cubic inches) in the trough when at these levels. Tabulate the results. (Recollect that the trough is only 32 inches long on the inside.)

(*d*) Draw a graph to show how the volume of water in the trough varies for different depths.

(*e*) If a gallon is 277 cubic inches, from the graph find the depth of water in the trough when it contains two gallons.

10. Some sheets of foolscap paper (13″ by 7·5″) make a pile 2·4″ high. The paper is moved in the direction of its length, so that the whole pile becomes a parallelepiped with a rectangular base. (The lowest sheet is unmoved, and the top sheet is moved most.) The move makes AB 5·1″ long. What is the volume and total area of the surface of the pile before and after the move ?



11. Steel tubing is to be made with external diameter 2 cm. It is to be of such a thickness that it weighs 1·5 Kgr. per metre length. The sp. gr. of steel is 7.

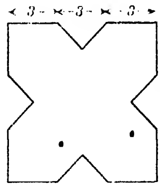
(*a*) Calculate the weights of tubes, 1 metre long and 2 cm. in (external) diameter, when the thicknesses are respectively 3, 4, 5 and 6 *mm.*

(*b*) Draw a graph to show how the weight varies with the thickness.

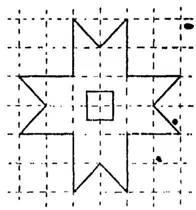
(*c*) From the graph determine the thickness (to the nearest tenth of a mm.) of the tubing if it is to weigh 1·5 Kgr. per metre length.

12. Cylindrical tins, to hold a commodity like condensed milk, are to be made to contain 500 cm³. [call this 160π cm³.], and it is desired to economise. Assuming the formula $V = \pi r^2 h$, calculate h when the radius is 2, 4, 6 and 8 cm. respectively, and thence calculate the total area of surface under these 4 circumstances. [keep π in the results]. Exhibit graphically the relation between the total area of surface and r. For what value of r is the total surface area a minimum? Calculate the value of h corresponding to this value of r. How do the diameter and height compare for the most economical tin?

* **13.** The cross-section of a metal bar is a square, 9 cm. by 9 cm., from the centres of whose sides right-angled isosceles triangles, with hypotenuses 3 cm., have been cut. The bar is 1 metre long, and the sp. gr. of the metal composing it is 7·5. What is the weight of the bar?



* **14.** The section of a hollow metal bar is shown. The sides of each of the dotted squares is 1 cm. (These are to show the dimensions.) The central square is of the same size. The bar is 1 m. long. What is (1) its volume and (2) the total area of its surface (both ends and inside included)?

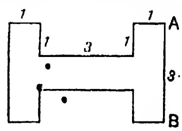


* **15.** The cross-section of a solid tyre is a rectangle whose sides are a inches and b inches, the former being the thickness of the tyre from road to wheel and the latter its breadth measured across the top of the wheel. If the outer diameter of the tyre is D inches, what is its inner diameter?

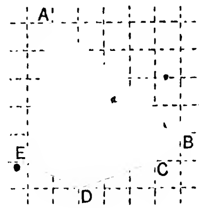
Assuming the formula for the volume of a cylinder, and considering the tyre as the difference of two solid cylinders, prove the formula $V = \pi ab(D - a)$ for the volume of the tyre in cubic inches.

Calculate D to the nearest tenth of an inch, having given $V = 600$; $a = 2{\cdot}3$; $b = 4{\cdot}2$.

* **16.** This symmetrical figure revolves round its side AB. Find the volume of the solid generated. [Dimensions in inches.]

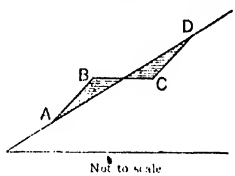


*17. ABCDE is the horizontal section of a block of wood. The dotted lines give its dimensions, for the side of each little square is 3 inches. PQRST is the top, correspondingly (PQ corresponds to AB, etc.). The top and bottom are exactly equal. The prism is sloping, T being vertically above C. The edges of PQRST are parallel to the corresponding edges of ABCDE. The slanting edges of the prism (ET for instance) are 25 inches long. What is the volume of the block? [The block is glued on to a horizontal board to prevent toppling.]

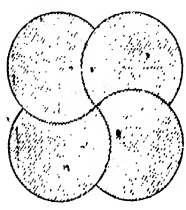


*18. A ditch is 90 ft. long and its cross-section is an equilateral triangle with 10 ft. sides. How many cubic feet of earth must have been taken out in making it; and what is the weight (to the nearest ton) of water in it, if it comes half-way up the sides?
[1 cu. ft. of water weighs about 1000 oz.]

*19. A hillside slopes at 1 in 5 (1 vertical and 5 horizontal). A horizontal surface for a lawn tennis court is to be made by excavating earth above and filling it below as indicated. BC is to be 20 yards. The slopes AB and CD are to be 1 in 2. By accurate drawing, find out how much must be excavated per yard length. If the total length excavated is to be 40 yards, and if a cubic yard of earth weighs $1\frac{1}{4}$ tons, find the tonnage of earth to be shifted.



*20. A pillar is made of portions of 4 cylindrical marble columns arranged (in section) as a square. The equal columns are just large enough to pass through the centre of the pillar. A tape measure, stretched tight, gives the girth as 7 feet. The pillar is 20 ft. high. 1 cu. ft. of marble weighs 170 lb. What is the weight of the pillar?



*** * 21.** OA, OB and OC are conterminous edges of a parallelepiped. OA $=6''$, OB $=7''$ and OC $=8''$.

$\angle BOC = 90°$, $\angle COA = 60°$ and $\angle AOB = 45°$.

What is the volume of the parallelepiped, and what is its surface area ?

Pyramids.

22. Draw a circle with radius 10 cm.

Calculate the circumference of the circle.

Calculate the area of the circle.

Cut it out and make a straight cut along one radius from circumference to centre.

Roll the paper round to make a hollow cone with two thicknesses of paper throughout. Fasten it with paste or gum or a couple of paper fasteners.

What fraction of the circumference of the original circle is the circumference of the base of the cone ?

And, hence, what fraction of the radius of the original circle is the radius of the base of the cone ?

What is this radius in cm. ? (Check by measurement.)

What fraction of the area of the original circle is the area of the circular base of the cone ?

What is this area in square cm. ?

What fraction of the area of the original circle is the area of the surface of the cone ?

Give this area in square cm.

Use Pythagoras' Theorem to calculate the height of the cone. (Check by measurement.)

Calculate the volume of the cone in cubic cm.

If a solid cone of copper of the same size weighs 2 Kgr., show that the sp. gr. of copper is about 8·8.

23. Make a "mathematical drawing" of a pyramid with a square base ABCD and vertex V; the edge of its base is 10 cm. and its height 12 cm. [Notice that in your drawing the vertex must be vertically above the middle point of the base.]

Draw the vertical section VLM, where L and M are the middle points of the edges AB and CD.

Calculate the length of the edge VA (to the nearest mm.), and check your result by drawing.

Calculate the volume of the pyramid, and the area of its total surface.

24. A pyramid stands on a square base whose sides are 10 cm., while the slanting edges are 13 cm.; find the height and volume of the pyramid.

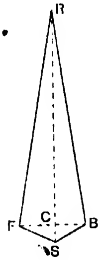
25. A pyramid stands on a square base of side 5·6 in.; if its height is 9·6 in., calculate its volume, and the total area of its surface. Find also the length of one of its slant edges.

26. A canvas conical tent, of capacity 600 cubic feet, is to be constructed on a circular base of area 160 square feet. Find, to the nearest square foot, the number of square feet of canvas required.

* **27.** VABC is a pyramid on a horizontal base ABC, and vertex V, with VA vertical, so that angles VAB, VAC are both right angles. BAC is also a right angle. VA = 15 cm., BA = 36 cm., AC = 20 cm. Calculate the volume of the pyramid. Also calculate the area of the total surface of the pyramid.

[*Note.* The *N.B.* in **41 c, 24** suggests a neat way of getting the area of the face BVC.]

* **28.** The figure represents a section of a "Minimax" Fire Extinguisher, called the 10-pint size. It may be taken as a hollow vessel (of thin metal) consisting of 2 cones, on the same circular base, whose diameter is FB. Dimensions in inches: RF = 23, CS = 1·3, and circumference of circular base, at FB, = 24. [1 gallon = 8 pints = 277 cubic inches about.]



Calculate FC (nearest tenth of an inch).

Draw the section RFSB to scale; measure RC.

Calculate RC (nearest tenth of an inch).

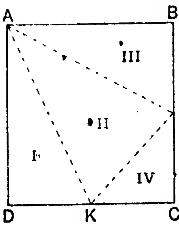
Calculate the total volume of the two cones (nearest cubic inch).

Find the corresponding number of pints (nearest tenth of a pint).

Calculate FS (nearest tenth of an inch).

Calculate the total area of the surfaces of the two cones (nearest square inch).

* **29.** ABCD represents a square of paper, K and L are the mid points of CD and CB, and the dotted lines show creases made by folding the paper along AK, AL and KL. A pyramid could be formed by folding the paper along the creases and bringing the edges AB, AD together, and also CK, CL against



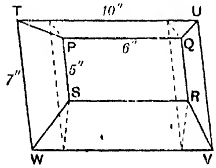
the edges DK, BL. If the length of the side of the square is 12 cm., calculate

(*a*) the lengths of the edges AK, KL;

(*b*) the areas of the separate faces I., II., III., IV.;

(*c*) the volume of the pyramid (you will observe that when the pyramid is formed the edge AB is perpendicular to the face KCL).

* **30.** A block of lead is bounded by six plane faces as shown in the figure. The base TUVW and the top PQRS are rectangular. The slant faces PSWT and QRVU are equal in all respects, and so are the faces RSWV and PQUT. Certain dimensions are shown in inches upon the figure. The height is 4 inches.



Imagine the block to be cut in three by vertical planes through PS and QR, the central piece to be removed and the two end pieces united with their cut faces coincident. Sketch the two resulting blocks, showing their dimensions.

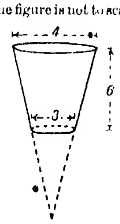
In the block made of the two end pieces P and Q coincide, and so do R and S. Imagine this block cut in three by vertical planes through P and S parallel to VW, the central piece removed, and the two end pieces united with the cut faces coincident. Sketch the two resulting blocks and show their dimensions.

The volume of a pyramid is one-third of the product of the base and the height. Use the blocks into which the lead has been rearranged to find the volume of lead in the original block.

* **31.** A pyramid (on a square base with sides 9 inches) is 8 inches in vertical height. The top is cut off by a horizontal plane which bisects the axis. If the pyramid is made of wood weighing 60 lb. per cubic foot, what is the weight of the frustum remaining?

* **32.** A tin cup is in shape the frustum of a right cone; the internal diameters of the cup at the top and bottom are respectively 4 inches and 3 inches, and the internal depth is 6 inches. Suppose a conical piece added to the cup, so as to complete the cone. Find, by calculation or by drawing, the height of the added cone and the height of the whole cone.

The figure is not to scale.



Find also how many pints the cup will hold.

* **33.** Find the volume of a hollow metal cone whose height is 4′ 8″ and the *diameter* of whose base is 2′ 8″.

If a *similar* cone were of height 4′ 1″, what would be the diameter of its base? Calculate the volume of this cone.

If this second cone were cut away from the first cone and the rim of metal left were made into a "saucer-bath" by the addition of a circular metal plate, find the volume of the bath in gallons. [1 c. ft. = about $6\frac{1}{4}$ gallons.]

* **34.** A sheet of zinc in the shape of a sector of a circle of radius 20″ and angle 120° is bent into a cone; the cone is cut in two at a point $\frac{2}{3}$ of the way down its height, as measured from the vertex; the larger piece is then formed into a bucket by fixing a circular plate to its smaller opening; calculate, as shortly as possible, the outside surface of the zinc bucket in square inches, and the number of gallons of water that it will hold, to the nearest whole number.

[A cubic foot of water and a gallon of water weigh respectively *about* 1000 oz. and *exactly* 10 lb.]

* **35.** The floor of a building is in the shape of a regular octagon with sides 10 ft. The walls are 8 ft. high. Then comes a pyramidical roof. The *total* height of the building is 14 ft. If people are to be allowed 500 cu. ft. of air space each, how many people (on a maximum) should it contain?

* **36.** A tin can, in the shape of a frustum of a cone, (like a railway milk can), has upper and lower circumferences 33″ and 55″ and is of depth 30″. Find its capacity to the nearest gallon. [1 gallon = 277·2 cu. in. roughly.]

* * **37.** A triangle, with sides 5″ and 6″ and included angle 120°, revolves round the 5″ side. Calculate the area and volume of the solid generated.

* * **38.** A triangle has sides 10″, 17″ and 21″. Calculate the length of the perpendicular, to the longest side, from the opposite angle. What is the volume of the solid generated by the revolution of the triangle about its longest side? [Leave π in the result.]

Also calculate the area of the triangle.

If the revolutions had been about the first two sides, in succession, show that the volumes would have been

$$\frac{9408\pi}{10} \text{ cu. in. and } \frac{9408\pi}{17} \text{ cu. in.}$$

* * **39.** ABC is a triangle in which $a = 3''$, $\angle B = 60^\circ$, $\angle C = 45^\circ$. On its sides BC, CA and AB respectively, and external to it, are described 3 triangles PBC, QCA and ARB, such that $QA = AR = 2\frac{1}{4}''$, $RB = BP = 2''$ and $PC = CQ = 2\frac{1}{2}''$. The last 3 triangles are bent about the sides of the first triangle ABC, and the points P, Q and R are brought together to a common point V. Find the volume of the tetrahedron VABC.

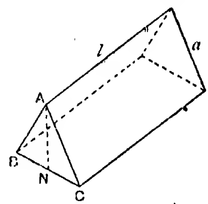
EXAMPLES 45 b (CALCULATIONS. GENERAL CASES).

Prisms.

O **1.** Into how many pyramids do the 4 diagonals of a parallelepiped divide it? What is the volume of one of those pyramids compared to the original parallelepiped?

2. A glass equilateral prism is of length l, and its equilateral triangular edges are a. Calculate its volume.

[HINT. Calculate the perpendicular, from A to BC, namely AN, in the $\triangle$ABC.]

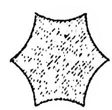


* **3.** The centres of the faces of a cube are joined to form a regular octahedron. The centres of the faces of that regular octahedron are joined to form a second cube. What is the volume of the second cube compared to the first?

* **4.** A right prism, of height h, stands on a base which is a regular hexagon of side a. Calculate (1) the total area of its surface and (2) its volume.

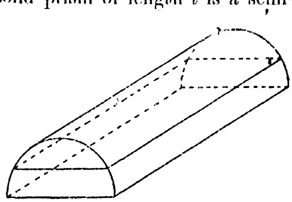
* **5.** The sides of a rectangle are a and b. It revolves round those sides in succession. Prove that the volumes of the cylinders generated are inversely proportional to those sides.

* **6.** A fluted column, whose diameter is $2r$, is leaning at an angle of 60°. The section as it issues from the (horizontal) ground is shown. (The corners are the angular points of a regular hexagon, whose sides are r, and the radii of the arcs are all r.) The *slope* length is l. The top of the column is horizontal. What is the volume of the column above ground?



* 7. The diagonals of the six faces of *any* parallelepiped are drawn and an (irregular) octahedron is formed by joining up their intersections. What is the ratio of the volume of the octahedron to that of the parallelepiped ?

* * 8. The cross-section of a solid prism of length l is a semi-circle of radius r. A plane, parallel to the flat face of the prism, and bisecting the radii which are at right angles to the diameters of the two semi-circles, divides the prism into two portions. Give the ratio of (1) the total areas of the two surfaces (including the flat faces) and (2) the volumes.



Pyramids.

O 9. Pyramids of the same volume are on the same base and on the same side of it ; what is the locus of their vertices ?

What is the locus if the pyramids may be on either side of the base ?

10. A trapezium with parallel sides a and b ($a > b$), and perpendicular distance between them h, revolves about the side a. What is the volume of the solid generated ?

11. In the preceding question, if $b > a$, what is the volume ?

12. The edges of a regular tetrahedron are a. Calculate (1) the total area of its surface and (2) its volume. Leave the results in a surd form.

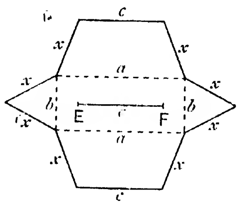
13. Calculate the area of the surface and volume of a regular octahedron whose edges are a.

14. VABC is a pyramid. A plane XYZ, parallel to the plane ABC, cuts VA, VB and VC respectively at X, Y and Z. Prove that the volumes of the pyramids VXBC, VYCA and VZAB are equal.

15. ABCD is any pyramid whose centre of gravity is G. Prove that the volumes of the four pyramids GBCD, GCDA, GDAB and GABC are equal.

16. Missing out 4 non-adjacent corners of a cube, show that the diagonals of the 6 square faces of the cube form a regular tetrahedron. If an edge of the cube is a, what is the volume of each of the 4 pyramids which can be cut off the original cube to leave this tetrahedron ? What is the volume of the tetrahedron ?

17. The accompanying figure, which is not drawn to scale, is symmetrical with respect to the line EF and with respect to the perpendicular bisector of EF. Imagine that the rectangle is the base of a tent bounded by four sloping walls, and that the parts of the figure outside the rectangle are turned about its sides to form the walls of the tent; the four walls terminate in a horizontal ridge which stands at a height h over the line EF. You may make a model from a sheet of paper. The volume V of air-space enclosed by the tent is given by the formula $V=\frac{1}{3}(2a+c)D$, where D denotes the area of the vertical section which bisects EF at right angles. Express V in terms of a, b, c, h.

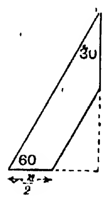


Let A denote the area of the base of the tent and M the area of the horizontal section which bisects the altitude of the tent. Obtain an expression for V in terms of A, M, h.

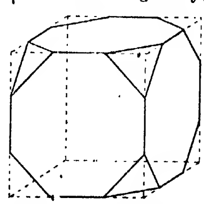
Also calculate V when $a=40$ ft., $b=24$ ft., $c=30$ ft., $h=16$ ft.

* **18.** Prove that the area of the curved surface of a frustum of a cone is equal to the slant height multiplied by the perimeter of the mid-section.

* **19.** A trapezium is shown in the figure. A right-angled triangle is made by producing the non-parallel sides. The production exactly doubles those sides. The trapezium revolves about the (vertical) side $\frac{h}{2}$. What is the volume of the solid of revolution described? Give the result in terms of h only.



* **20.** The eight vertices of a cube are truncated regularly, leaving a regular octagon on each of the faces, which were originally square, and equilateral triangles at the corners. What is the area of the surface and the volume of this solid, both compared to the original?

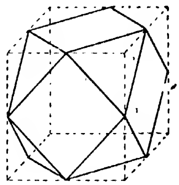


The figure, in which only 3 of the octagons and 3 completed equilateral triangles are shown, gives the idea.

Calling the side of the octagon s and of the cube a, what is the relation between s and a?

[The solid is called a *truncated cube*.]

* **21.** The middle points of each of the 12 edges of a cube are joined to leave an equilateral triangular section at each of the 8 corners of the cube; and a square on each of the 6 faces of the cube. [The figure shows the plan, but only some (of the many) lines are shown.] The edges of the cube are a.



The 8 corners of the cube are cut away. What is the volume of one of the 8 pieces cut away? What is the volume of the solid left?

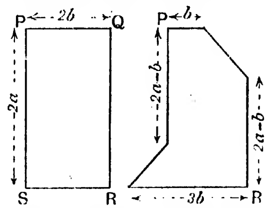
How many square faces has it got? How many triangular faces has it got? What is the total area of its surface?

[The solid is called a *cubo-octahedron*.]

* **22.** A *truncated octahedron* is formed by truncating the vertices of a regular octahedron, so as to leave the original faces regular hexagons; consequently, it is bounded by 8 hexagonal and 6 square faces. In what ratio are the edges of the original solid compared to those of the regular hexagons? What is the area of a hexagon compared to the face of the octahedron on which it is? What is the total area of the surface of the solid compared to that of the original? What are the comparative volumes?

[HINT. (For the last part of the question.) What is the volume of each little pyramid cut off, compared to half the original octahedron?]

* **23.** State in what ratio the height of the pyramid must be divided by a plane parallel to the base, in order to divide the pyramid into two parts whose contents are equal to one another.



* * **24.** (1) A rectangle PQRS (in which $PS = 2a$ and $PQ = 2b$) revolves round the side QR.

(2) An isosceles triangle, with equal sides b, is cut off the corner Q, and fixed on to the corner S, and the

new irregular hexagon revolves about the same side as before.

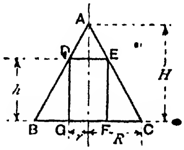
[It is given that $a > b$.] By how much does the volume of the second solid exceed that of the former, if there is a complete revolution in each case?

** **25.** Referring to the figure of the previous question, by how much is the area of the surface of the Solid of Revolution increased by the transfer of the small triangle from the corner Q to the corner S?

** **26.** DEFG is a cylinder of radius r and height h inscribed in a cone of radius R and height H.

Show that the volume of the cylinder can be expressed in the form

$$V = \frac{\pi R^2}{H^2}(H^2h - 2Hh^2 + h^3).$$

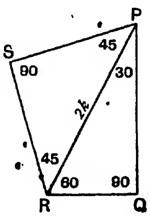


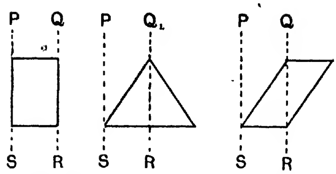
If R and H are both equal to unity, find the values of V when $h = 0$, 0·1, 0·3, 0·5, 0·7, 0·9, 1; plot the graph showing the relation of V and h. State from the graph the approximate value of h which makes V a maximum.

** **27.** A conical extinguisher, whose section through the vertex is an isosceles triangle with vertical angle 30°, is placed over a cylindrical candle whose diameter is 2 cm., and rests so that the point of contact of the candle with each generating line of the cone bisects that line. Find the whole inside surface of the extinguisher.

** **28.** It is required to divide a solid right cone into two portions by a plane parallel to the base, so that the surfaces of the portions (including the flat faces) may be equal. Find the ratio into which the slant sides must be divided by the plane.

** **29.** Two set-squares, with equal hypotenuses PR, are put side by side to form the single quadrilateral PQRS indicated. The whole revolves about the line PQ for one revolution. If the equal hypotenuses are $2k$, find the volume of the solid described.





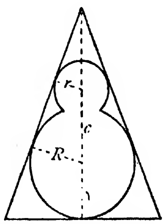
* * 30. (1) A rectangle, 4 by 3, revolves about one of its longer sides for a complete revolution (*i e.* either about PS or about QR). Find the total area of the surface and volume of the solid of revolution so formed. [Leave π in the answers.]

(2) The rectangle is cut in half, along a diagonal, and the two halves are joined into one isosceles triangle. The figure is revolved about PS, or about QR. What are the areas and volumes of the solids of revolution? [Leave π in the answers.]

(3) Another junction is indicated, the two halves being joined into one parallelogram. Revolutions about PS, or about QR, are performed. What are the 4 answers now? [Leave π in the answers.]

Tabulate all your answers.

* * 31. The centres of two spheres, with radii R and r, are c apart. Find the volume of the least cone, which contains them.



* * 32. A rhombus revolves first about one diagonal, and secondly (separately) about the other diagonal. Prove that both the volumes of the two solids generated, and their surfaces, are inversely proportional to the length of the diagonal about which the revolution takes place.

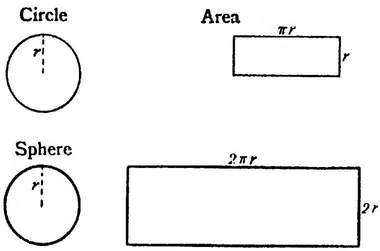
* * 33. A triangle, with sides a, b and c, revolves once (separately and in succession) about each of its three sides. If Δ is the area of the triangle, what are the volumes of the three solids generated?

CHAPTER XLVI.

AREAS AND VOLUMES (SPHERES AND SPHERICAL SEGMENTS).

§ 1. *It is difficult to realize (from a plane diagram representing solidarity) that the area of the surface of a sphere is four times the area of the equator circle.* Perhaps the figure, which is from a photograph, showing a lawn tennis ball and a fives ball, with their covers on and off, may help you to realize this fact, and to make the following more intelligible. [IT SHOULD BE CLEARLY UNDERSTOOD THAT THE ILLUSTRATIONS MAY EMPHASIZE A POINT, BUT THEY ARE CERTAINLY NOT PROOFS.]

The area of the surface of a sphere is equal to that of a rectangle, whose sides are equal to a diameter and circumference of a 'great' circle of the sphere.



The following comparison should be noted:

(1) **Area of a Circle = semi-circumference × semi-diameter.**

(2) **Area of surface of Sphere = whole circumference × whole diameter.**

* § 2. The figure represents a section of a **sphere and the circumscribing cylinder** through the axis of the latter.

P and Q are points on the surface of the sphere, on this section. (Later on P and Q are going to be *very* close to each other.) K and L are the corresponding points on the circumcylinder. PN is perpendicular to LQ.

We are going to consider the band (or narrow zone of the sphere), whose length is 2π. PY and width is PQ.

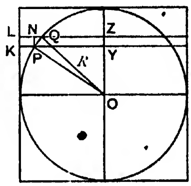
[Notice P and Q are ultimately going to be made to coincide, so that it is quite immaterial whether we call the radius of the band PY or QZ or some sort of average between them.]

Now P and Q are *very* close, and when they are made to coincide, the chord PQ becomes a tangent;

then $\angle QPO = \text{rt. } \angle$,

but, $\angle NPY = \text{rt. } \angle$;

from each subtract $\angle QPY$;



$\therefore \angle YPO = \angle NPQ$,

and $\angle PYO = \angle PNQ$ (rt. $\angle$s):

$\therefore \triangle YPO$ and $\triangle NPQ$ are similar:

$\therefore PY : PO = PN : PQ$;

hence $PY : KY = KL : PQ$;

$\therefore PY \,.\, PQ = KY \,.\, KL$;

$\therefore 2\pi PY \,.\, PQ = 2\pi KY \,.\, KL$.

By taking the next, and the next, and the next, ... etc., bands and adding, we get at the fact,

that the **Area of a Zone of a Sphere = area of the corresponding zone of the circumcylinder.***

Whence **Area of Sphere = area of curved surface of circumcylinder.**

Now the area of the curved surface of circumcylinder

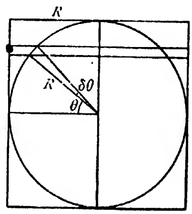
$$= 2\pi RH = 2\pi R \times 2R = 4\pi R^2;$$

so that **Area of Sphere** $= 4\pi R^2$.

* It is to be noted that a zone of a sphere includes the case of a **cap of a sphere.**

The following section should be considered as an alternative to the preceding.

*** * § 3. A formal proof of the formula for the area of the surface of a Sphere, using the Integral Calculus, follows.** [Of course, there are various ways of getting round this notation, but they are all much the same. It is better to learn that powerful subject than various tricks for avoiding it.]



Radius of band of sphere $= R\cos\theta$.

Circumference of same $= 2\pi R\cos\theta$.

Width of same $= R\,\delta\theta$.

$$\therefore \text{ Area of sphere} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\pi R^2 \cos\theta \,.\, d\theta$$

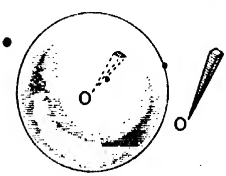
$$= 2\pi R^2 \Big[\sin\theta\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= 4\pi R^2;$$

$\therefore$ **Area of the surface of a Sphere $= 4\pi R^2$.**

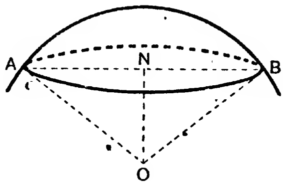
[*N.B.*—For Zones the proof is substantially the same; it is only necessary to work with appropriate "Limits."]

§ **4.** For the **volume of a sphere** imagine a multitude of pyramids, fitted together, having their vertices coincident at its centre O, and having their (any-shaped, but very small) bases covering the whole surface of the sphere. Now the radius of the sphere is equal to the height of any one of those pyramids, when their bases are made infinitely small. Hence it follows that the volume of a sphere is equal to the area of its surface $\times \frac{1}{3}$ of its radius, or, briefly,



Volume of a Sphere $= \frac{4}{3}\pi R^3$.

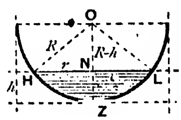
* § **5.** In the same way as we consider the *area* of a segment of a *circle* as the difference of the areas of a sector and a triangle, so the **volume of a spherical segment** is considered as the difference of the volumes of a spherical sector and a cone.



For instance, the volume of the spherical segment (the diameter of whose base is AB) is equal to that of the spherical sector (whose centre is O and base-diameter AB) minus the cone (with O as vertex and with the same base).

§ **6.** The following is an **Example** of calculation.

A hemispherical wine-glass is filled to $\frac{2}{5}$ depth. Show that it is but little over $\frac{1}{5}$ full.



The spherical segment HZL = spherical sector OHZL minus the cone on circular base HL and height ON

$=$ spherical cap HZL $\times \frac{1}{3}R - \frac{1}{3}\pi \text{HN}^2 \,.\, \text{ON}$

$=$ area of zone of a circumcylinder $\times \frac{1}{3}R$
$- \frac{1}{3}\pi r^2(R-h)$

$= 2\pi Rh \times \frac{1}{3}R - \frac{1}{3}\pi r^2(R-h)$;

and, by simplification, this reduces to $\frac{1}{3}\pi h^2(3R-h)$.

Now the volume of the hemispherical glass is $\frac{2}{3}\pi R^3$; and, in this case,
$$h = \frac{2}{5}R;$$
whence, by substituting, the fraction filled is
$$\frac{\frac{1}{3}\pi \times \frac{4}{25}R^2 \times \frac{13}{5}R}{\frac{2}{3}\pi R^3},$$
which reduces to $\frac{26}{125}$, and this is little over $\frac{25}{125}$ or $\frac{1}{5}$.

*§ 7. If an area revolves round an axis it generates a **Solid of Revolution**. If the area is bounded by straight lines, then cylinders, cones (including frusta of cones), or definite mixtures of them, are described; and many have already been considered in the preceding chapter. If the area is partly, or wholly, bounded by circular arcs, and if the axis of revolution passes through the centres of those arcs, the addition (to the foregoing) of spherical segments will complete the possible shape of the solid. To find the area of their surfaces, and their volumes, is thus only an extension of the preceding.

[Familiar examples are the following, which may be more or less perfect, a bell, many products of a potter's wheel and of a lathe. Much fruit may be somewhat of this kind.]

EXAMPLES 46 a (CALCULATIONS. SPECIAL CASES).

1. "A cricket ball shall weigh not less than five ounces and a half, nor more than five ounces and three quarters," . . . and "it shall measure not less than nine inches nor more than nine inches and a quarter in circumference."

Taking the average weight and the average girth, what is the weight in ounces per cubic inch ?

2. Find, in millions of square miles, the areas of spheres with radii 0, 1, 2, 3 and 4 thousands of miles, and exhibit graphically the connection between the area of a sphere and its radius.

Use the graph to fill up a table like the following:

Planet.	Radius in miles.	Approximate area of surface in square miles.
Mercury	1488	
Venus	3814	
Earth	3958	
Mars	2158	

The radius of Saturn is about 9 times the radius of the earth. Give the area of Saturn's surface to 2 significant figures.

3. A cubical cardboard box just contains a sphere (like a single golf ball, without paper, fitting tightly into a box). What percentage of the box is empty? [Give the result to the nearest whole number.]

4. A spherical lump of ice with diameter 10 cm. melts so that it remains always spherical in shape, while its radius decreases at the rate of 1 centimetre in 10 minutes.

Draw a graph to show how the volumes of spheres vary with their radii (and thus how the volume of ice changes with respect to time).

[Take radii 0, 1, 2, 3, 4, 5 cm. to fix points for the graph.]

From the graph determine after what time the volume of the ice will be one-third of what it was originally.

[You may assume that the volume of a sphere equals $4{\cdot}2 \times (\text{radius})^3$.]

5. A sovereign's worth of half-crowns is melted down and cast into a single spherical ball. If the half-crowns are assumed to make a single solid cylinder of height 0·67″ and diameter 1·28″, what is the diameter of the ball? [Give the result to the nearest hundredth of an inch; for, at most, the data do not warrant a greater pretence to accuracy, and the relief of a real half-crown is ignored.]

6. A tall cylindrical glass vessel has a capacity of somewhat over a litre. The internal diameter is 7 cm. It is to be graduated throughout (up to 1000 cu. cm.) to show differences of 10 cu. cm. How far apart must the graduation marks be made?

Into it water is poured up to the 530 cu. cm. mark, and 105 brass spheres of diameter 2 cm. are put into the vessel. To what mark will the water rise? [Recollect that the graduations show differences of 10 cu. cm., and in consequence you need volumes correct to 10 cu. cm.]

7. A piece of soap, spherical in shape and $5\frac{1}{4}$ in. in diameter, dissolves in a bath until its diameter is $3\frac{1}{2}$ in., keeping spherical all the while. What volume of soap is dissolved, and what is the area of the piece of soap after immersion?

What would be the size of a spherical cake of soap made out of the soap which has been wasted?

8. A hollow spherical iron shell is to have an external diameter of 10 cm. and is to weigh 3 Kgr.

The sp. gr. of iron is 7.

(*a*) Taking the outside diameter as 10 cm., find the volumes of shells which have thicknesses 1, 2, 3 and 4 cm. respectively. (Do not substitute for $\frac{4}{3}\pi$.)

(*b*) Taking π as $\frac{22}{7}$, find the weights of those shells in grammes.

(*c*) Draw a graph to show how the weight of the shell varies with the thickness of the shell.

(*d*) Use the graph to determine what should be the thickness of the shell if it is to weigh 3 Kgr.

9. An Association football should be truly spherical with a girth between 27 and 28 inches. Does a ball of average girth contain more or less than $1\frac{1}{4}$ gallons? [1 gallon = 277 cu. in. about. Use logarithms.]

10. Two spherical soap-bubbles with the same thickness of surface, and with diameters 2″ and 4″, combine and make a single spherical soap-bubble with thickness of surface $\frac{4}{5}$ that of the original. What is its diameter?

* **11.** Two spherical iron shells X and Y are made:

X has external diameter 10 cm. and the metal is 2 cm. thick.
Y has external diameter 12 cm. and the metal is 1 cm. thick.
The sp. gr. of iron is 7.
If both are empty, which is the heavier and by how much?
If both are full of water, which is the heavier and by how much?

* **12.** Find the radius of a spherical balloon made of 2300 sq. ft. of skin: find also the weight of the gas, given that it is 0·0000895 times as heavy as water. [Take the weight of a cubic foot of water to be $62\frac{1}{2}$ lb.]

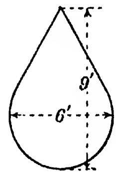
* **13.** A cubical box with a lid will just contain a solid sphere of radius 8″. Find the volume of the box, of the sphere and of the vacant part of the box.

The sphere is cut into eight equal sections through its centre, so that each section fits into a corner of the box where it is fixed. Find the distance between the nearest points of two diagonally opposite sections, and hence find the radius of the largest sphere which can be fitted inside the eight sections. (*N.B.*—The diagonal of the box can and should be shown to be $\sqrt{768}$ inches.)

* **14.** The distance of the Sun is more than 90 million miles, and the radius of the Earth is less than 4 thousand miles. Roughly, what fraction of the Solar radiation is received by the Earth?

[Hint. The distance of the Sun is so enormous compared to the radius of the Earth, and the data are clearly rough too, that it does perfectly well to consider that the Earth appears, to the Sun, a *circle* of 4000 miles radius.]

* **15.** A hollow buoy is made of sheet steel, and its dimensions are those shown.



The base of the buoy is part of a sphere and the upper portion a cone which touches the sphere.

The steel is $\frac{3}{8}$ inch thick and has a density of 480 lb. per cubic foot.

Assume that the dimensions are "mean" values between the internal and the external, and find the weight of the buoy.

* **16.** Suppose that roughly the radius of Mars is 2000 miles, and that a cap of snow stretches all round from the pole 40°. Find the approximate area of this cap.

* **17.** The covering of an umbrella forms a portion of a sphere of 21 in. radius, the area of the cover being 1600 sq. in. Find roughly the area of the ground sheltered from vertical rain when the stick is held upright.

Give the result in square feet to one significant figure (and no further).

* * **18.** A muddy Association football (circumference $27\frac{1}{2}$ inches) is kicked against a wall, and leaves a circular mark of diameter 5 inches. How much was it compressed? What was the volume before impact and the minimum volume?

* * **19.** A spherical melon stands 20 centimetres high, and consists of inedible rind (1 cm. thick) outside, and an empty spherical space inside of radius 4 cm.: the rest is edible.

(*a*) What volume is edible?

(*b*) How much per cent. is this of the whole volume?

(*c*) If the specific gravity of the rind is 0·8, and of the edible part 0·9, what is the weight of the melon?

(*d*) If it is cut into 10 slices, what is the total area of the surface of 1 slice?

* * **20.** A hemispherical bowl, weighing 280 grammes, and with diameter 12 cm., is floating on water. To what depth is it immersed?

[*N.B.* Find an equation to determine this depth. Let it contain π. It should be a cubic equation. Clearly only an approximate answer is appropriate. So give π an approximate value. Give the depth to the nearest half cm.]

EXAMPLES 46 b (CALCULATIONS. GENERAL CASES).

1. A hemispherical lump of putty is squashed into a circular disc, the diameter of the disc being double that of the hemisphere. If r is the radius of the hemisphere, calculate the thickness of the disc.

2. A boiler consists of a cylindrical body with hemispherical ends: show that its volume may be expressed in the form $\frac{\pi r^2}{3}(3l-2r)$, where l is its total internal length and r its internal radius.

Further, find the necessary internal length for a boiler of this shape and radius 50 cm. if it is to hold 2000 litres.

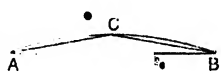
3. A sphere is inscribed in a cylinder. Prove that

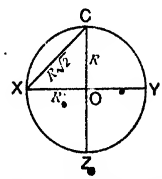
(1) the volume of the former is $\frac{2}{3}$ that of the latter,

(2) the area of the surface of the former is *also* $\frac{2}{3}$ the total area (including the flat ends) of the surface of the latter.

[**Archimedes** (roughly 250 B.C.) discovered this. It is said that he directed his friends and relatives to place upon his tomb a representation of a sphere and circumscribing cylinder, together with an inscription giving this ratio. Doubtless he regarded this discovery as the greatest of his very many achievements. His mathematical genius has perhaps not been surpassed.]

4. ACB is a cap of a sphere. A and B are the ends of a diameter of its plane surface, and C is the central point on its curved surface. Show that the area of the curved surface of the cap is equal to the area of a plane circle whose radius is equal to the chord CA.*

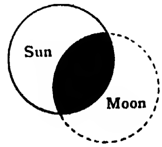




[In particular, notice that the area of the curved surface of a hemisphere is equal to the area of a (*plane*) circle whose radius is CX. Again the area of the surface of a sphere is equal to the area of a (*plane*) circle with radius CZ. These are only special cases of the general theorem above.]

* **Established by Archimedes.** See note on preceding question.

* **5.** The Sun and Moon are approximately *spherical*; but, being so far from the Earth, appear to be as *circles* in the sky. It happens that their distances are such as to make them appear to be about the same size.



Suppose that they are apparently exactly equal in size, and that, at a certain phase of an eclipse of the Sun, the edge of the Moon appears to pass through the centre of the Sun. What decimal of the apparently circular disc of the Sun is obscured? Give the result correct to one significant figure and no more.

* **6.** How far from the surface of a sphere must an eye be to see $\frac{1}{y}$ of its surface?

Using this result, draw a graph to show the positions of the eye, so that $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, etc., of its surface may be seen.

* **7.** A cube, with edge a, is bounded by a surface made of a membrane which can assume any shape, but which must maintain the same thickness always.

The cube is blown (like a soap bubble) into a spherical shape, the area of its surface being unaltered.

What is the diameter of the sphere? Is it more or less than $\frac{1}{3}$ greater than the edge of the cube?

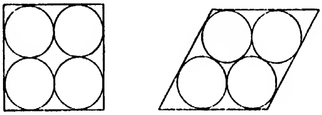
* **8.** A sphere and a cube have the same volume. Express the diameter of the former as a percentage of the edge of the latter. Give the result as a whole number, and no further. To the same degree of accuracy, compare the areas of the surfaces of the originals, taking that of the cube as 100.

* **9.** The floor of a box is completely covered with equal marbles. Above is a layer, of exactly the same size (so that the marbles are vertically above those in the layer below, and do not sink into the interstices between marbles). A third layer follows. What is the diameter (compared to that of a marble) of the largest ball-bearing that could trickle down from the top to the bottom of the box? If water were poured into the box, until its surface were level with the top of the top layer, what is the percentage of the volume of water and the volume of marble in the whole box? [Give the former result in surd form, and the latter to the nearest whole number. *N.B.*—The marbles are arranged squarely and not hexagonally.]

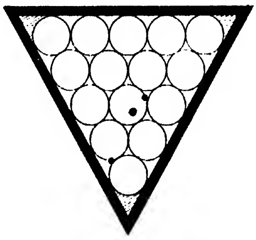
10. * * Show that the area of the surface of the zone of a sphere, between latitudes λ and ϕ, is given by $2\pi R^2(\sin\phi - \sin\lambda)$, where R is the radius of the sphere.

Draw a graph to show the variation in the area of belts, of width 10° of latitude. From the graph determine the latitudes of the boundaries of a belt of 10°, whose area is $2\pi R^2(0{\cdot}085)$.

11. * * Show that the volume of a cap of a sphere, between latitude λ and the pole, is given by $\frac{1}{3}\pi R^3(1 - \sin\lambda)^2(2 + \sin\lambda)$, where R is the radius of the sphere.



12. * * Four equal spheres are arranged either (1) as a square or (2) as the rhombus shown. A box, with plane surfaces, and of height equal to the diameter of a sphere, is made to contain them. If r is the radius of a sphere, give the volumes of the boxes in each case. Which is the larger? Give the excess of the volume of the larger over the smaller as a percentage of the smaller, correct to one significant figure.

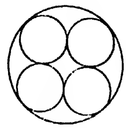


13. * * An equilateral triangular frame, strengthened at each of its corners by angle-pieces, just holds 15 equal balls for the game of *Pyramids*. The height of the frame, and of the angle-pieces, is exactly equal to the diameter of a ball. What percentage of the frame is empty? Give the result to the nearest quarter.

[*N.B.*—Both π and $\sqrt{3}$ come in. Do not substitute any numerical values for these, until the very end.]

* * **14.** A closed cylindrical box, to contain 4 equal metal spheres of radius r, is to be made.

(1) The first plan adopted is to have all four spheres arranged as a square, as indicated. The height of the box is exactly equal to the diameter of a sphere. What is the total area of the surface of the (closed) box ?



(2) The second plan adopted is to put three spheres as an equilateral triangle, and then to make the cylindrical box just tall enough to allow for the 4th sphere resting on these. What is the total area of the surface then ?

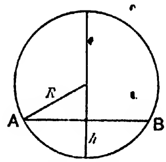
(3) In the third plan, the 4 spheres are melted up and cast into a single cylinder, whose height and diameter are the same. What is the total area of the surface of the box which would contain it ?

Representing the material needed for the box, in the first case by 100, express (to the nearest whole number) the amount needed in the other two cases.

[The thickness of the material is the same in all cases.]

Spherical Segments.

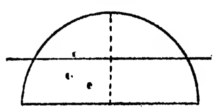
* **15.** A sphere, with radius R, is divided by the plane AB into two spherical segments. The height of the smaller is h. What is the ratio of the volumes of the two segments ?

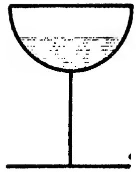


* **16.** In the preceding example what is the ratio of the total areas (including the plane face in each case) of the two segments ?

* **17.** A spherical segment (like a plano-convex lens) is indicated in the figure. Find its volume in terms of h and r.

* **18.** The radius, which is perpendicular to the plane face of a hemisphere, is bisected at right angles by a plane parallel to the former. What is the ratio (1) of the areas (including the plane faces) of the two portions and (2) of their volumes ?





* **19.** A hemispherical wine glass is filled to half depth. Show that is less than $\frac{1}{3}$ full, as far as volume is concerned. What is the exact fraction of its volume that is full ?

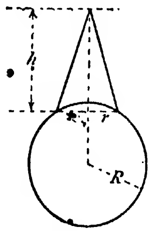
* **20.** A thin saucer, of diameter $2r$, has a maximum depth k. This saucer is a segment of a sphere. Calculate the area of its surface (inside and outside). It is filled to the brim with water. What is the volume of the water ?

[HINT. First calculate the radius of the whole sphere, of which the saucer forms a part.]

* **21.** If r is the radius of the plane face of a plano-convex lens (see the figure of Question 17), and h its maximum thickness, what is the total area of its surface ?

* * **22.** Every generator of a cone, with semi-vertical angle 60°, touches a sphere of radius r. What is the volume between the two ?

* * **23.** A square, ABCD, has sides r. A and C are the centres of two equal spheres with radius r. What is the volume of the portion common to both spheres ?



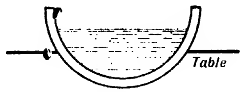
* * **24.** A hollow cone, of height h and base radius r, has an open mouth, and is put, like a dunce's cap, on to a sphere of radius R.

What is the volume contained between the two ?

[*N.B.*—It is to be assumed (as is indicated in the figure) that the cap is too small to fit over the sphere (forming a tangent cone to it).]

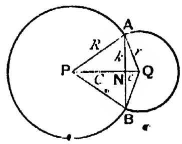
* * **25.** Dry sand issues from a small aperture and falls on to a horizontal table, when a cone of height h, and base-radius r, is formed. This sand is swept into a hemispherical bowl of radius R. It is shaken so that the top surface is flat, and then it is found that the maximum depth of sand is k. Show that $k^2(3R-k)=r^2h$.

* * **26.** A hemispherical washing-basin, with radius (external) R and thickness k, rests in a circular hole, of radius r, in the washing-table. Water, to a maximum depth h, is poured into the basin. What is the height, above the table, of the level of the surface of the water ?

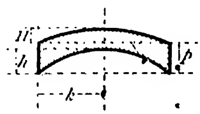


* * **27.** Two spheres have joined, like two soap bubbles coalescing. Their centres are b apart and the radii of the (*incomplete*) spheres are R and r. What is the volume ?

[HINT. Call PN $= C$ and QN $= c$, AN $=$ BN $= k$. These letters are to be left in the answer. It is perfectly possible to eliminate these letters and to give the answer in terms of b, R and r only ; but the general value is not neat.]



* * **28.** Determine the volume of the *concavo-convex* lens shown. The thickness of the lens (at the outside) is p. The whole lens is cylindrical in shape with radius k. Other dimensions are indicated also.

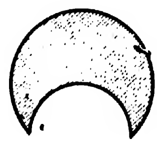


* * **29.** A lune, shaded in the figure, is described.

The radius of the convex semicircle is r, the distance between the horns is $2r$, the mid point of that distance is the centre of the (convex) semicircle. The radius of the concave arc is $r\sqrt{2}$.

The whole is revolved about its axis of symmetry. What are the area and volume of the solid of revolution ?

* * **30.** The shaded lune revolves round its axis of symmetry (the line bisecting perpendicularly the join-of-the-horns-of-the-lune, which is $2r$ long). The radius of the larger circle is $r\sqrt{2}$, and its centre is on the semicircle, whose radius is r. Find the volume of the solid generated..



* * **31.** Water, to a maximum depth k, is in a hemispherical basin of diameter $2R$. A cylindrical glass, of radius r, floats in the water. The glass is weighted with water, so that the glass is immersed vertically to a depth h. [$h <$ height of glass.] Show that the rise of the water line, in the basin, is given by $K - k$, where

$$K^2(3R - K) = k^2(3R - k) + 3r^2h.$$

CHAPTER XLVII.

AREAS AND VOLUMES (RELATIVE).

§ 1. In this chapter the Areas of the Surfaces and the Volumes of Solids are considered **relatively.**

Now, of course, all these are really relative. When we say that the volume of a body is 3 cubic inches, we are measuring its volume relative to the well-known cubic inch (derived directly from the British standard yard, established by law). When the comparison is not made with a legal standard, in such a case as "This glass holds twice as much as that," they are classed as relative in this chapter.

§ 2. The same properties, in regard to magnitude, which belong to solids bounded by plane faces, apply equally well to solids bounded by curved faces. For proof it only means that the bases of the pyramids (into which we can imagine them decomposed) must be taken infinitely small, and that there must be an infinite number of such.

Thus **the Areas of the Surfaces of similar solids vary as the square of their linear dimensions, and the Volumes of similar solids vary as the cube of their linear dimensions.**

In gauging areas and volumes, especially from models, it is important to get these two facts clear.

§ 3. You should notice that, in comparing the magnitudes of **two things, similar in shape, it is not necessary (and generally unduly laborious) to find their sizes separately.**

§ 4. As an **instance** of Relative Areas and Volumes, consider the following:

An exact model is made of a room and its contents, on the scale of 1 : 10. Notice that the poker length, the carpet width, the door height, etc., are all reduced in the ratio 1 to 10 (for, it is a question of 1 dimension in each case). However, the area of the table-cloth and the amount of paper for the walls are reduced in the ratio 1 to 10^2, or 1 to 100 (both being a question of 2 dimensions). Again, the volume of the air, the weight of one of the chairs, the amount of coal in the coal-scuttle, etc., are all reduced in the ratio 1 : 10^3, or a thousandfold, (a question of 3 dimensions in each case).

§ 5. For **Example** the diameter of the Sun is about a *hundred* times the diameter of the Earth.

Hence the volume of the Sun is about a *million* times the volume of the Earth.

$$[100^3 = 1{,}000{,}000.]$$

[To realize how enormous the Sun is compared to the Earth, you might know that, if the Earth were situated at the centre of a hollow sphere of the same size as the Sun, there would be ample room (and very much to spare) for the Moon to circulate round the Earth, in her proper orbit, all inside the hollow sphere.]

EXAMPLES 47 (CALCULATIONS. SPECIAL CASES).

O **1.** There are two spheres, and the diameter of the former is twice the diameter of the latter. In what ratio are (1) the areas of their surfaces and (2) their volumes?

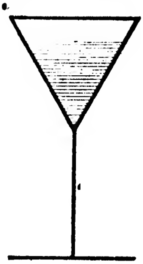
O **2.** Extracts from laws of Cricket and Football: "A cricket ball shall measure not less than nine inches or more than nine inches and a quarter in circumference." "An Association football must be a perfect sphere from 27 to 28 inches in circumference." Taking the circumference of the latter to be exactly three times that of the former, what are their comparative volumes?

O 3. A cubical box (like a very large biscuit box) has edges each 9 inches long.

Either a single sphere (of diameter 9 inches) is put in, or the box is packed full with little spheres (each of diameter 1 inch), with 9 layers (there being 81 of these little spheres in each layer). If the big and little spheres are of the same material, which is the heavier load ?

4. Each of the six edges of any tetrahedron is trisected ; and from each of its four corners a little tetrahedron is cut off, by planes through the three points of trisection adjacent to the corner. What is the volume of the solid left, compared to the original ?

5. A metal sphere is of radius 10 cm. ; if this is melted down and recast into spheres each of radius 1 cm., how many of these small spheres can be made ? In what ratio will the total surface be increased ?



6. The depth of wine in a conical wine-glass is $\frac{4}{5}$ of the depth of the glass. Show that the glass is but little more than half-full.

7 Two similar electro-plate cups are made of heights 4 and 5 inches. Show that the volume of the former is but slightly more than half the volume of the latter. If they are plated to the same (and not proportional) thickness, what is the percentage of the plating of the former compared to the plating of the latter ?

[There is no reason whatever to suppose that the cups must be cylindrical. They need not be. You should notice that the problems proposed in the first part of this question and the preceding question, mathematically, are the same.]

8. Three metal globes, of diameters 3″, 4″ and 5″, are put into a bath containing water, and are totally immersed ; the rise in water level is noted. They are taken out and a single metal globe, of diameter 6″, is put into the bath ; again it is totally immersed. Compare the rise in the water level in the first and second cases.

9. Three leaden spheres, of diameters 3″, 4″ and 5″, are melted up and cast into a single sphere. What is its diameter ?

10. Three spherical soap bubbles, with diameters 6 cm., 8 cm. and 10 cm., combine to make a single spherical soap bubble *with volume unchanged.* What is its diameter? What is the ratio of the total surface area before and after?

11. Two spherical soap bubbles, with diameters 3 cm. and 4 cm., combine to make a single spherical bubble, *with surface area the same as that of the total of the other two.* What is its diameter? Also what is the ratio of the sum of the original pair of volumes to the final volume?

12. A model, on the scale 1 : 10, is made of an engine. If the model weighs 5 lb., what would be the weight of a model of the same engine on the scale 1 : 9? Give the result to the nearest ounce.

13. A billiard ball (with diameter $2\frac{1}{16}''$) and a pyramid ball (with diameter $2''$) are made of exactly the same sort of material. Express the difference of their weight as a percentage of the weight of the larger, correct to one significant figure (and no further).

14. An exact model is made of a ship, on the scale of $\frac{1}{2}$ inch to a foot. In what ratio are lengths, areas and volumes respectively reduced? If the area, to be painted, of one funnel of the model is 165 square inches, what area of paint is necessary for the corresponding thing on the real ship?

* **15.** Two sticks of sealing-wax are precisely similar in shape, but different in size. Of the larger, 10 go to the lb., while of the smaller 40. Give the length of the smaller as a percentage of the larger, to the nearest whole number.

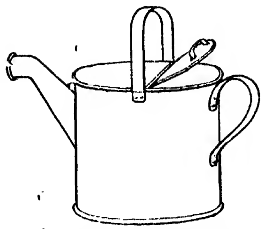
* **16.** A golf ball of diameter $1{\cdot}62''$ weighs $1{\cdot}62$ oz. If the diameter of a similar ball were $1{\cdot}65''$, what would be the weight? [Give the result to 3 significant figures, and no further.]

* **17.** "An Association football must be a perfect sphere from 27 to 28 inches in circumference." By how much per cent. is the volume of the maximum ball greater than the volume of the ball of average circumference? Also give the extra area of the case as a percentage.

Is the volume of the ball of average circumference more or less than the average volume of maximum and minimum balls?

* **18.** "A cricket ball shall measure not less than nine inches or more than nine inches and a quarter in circumference." By how much per cent. is the volume of the maximum ball greater than the volume of the ball which has the average circumference? Give the result to one significant figure.

*** 19.** Three similar hot-water cans have their lengths in the ratio 8 : 9 : 10. What is the ratio of the area of metal composing each ? Also what is the ratio of their capacities ? Express that of the largest as 100, and give each to the nearest whole number.



*** 20.** There are 3 hot-water cans of exactly the same shape. [See figure of Question **19**.] The sizes are 4, 6 and 8 pints. Give the ratio of their comparative lengths, taking 100 as the length of the greatest. What is the ratio of the comparative areas of their surfaces, taking 100 as the area of the greatest ?

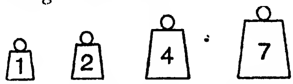
*** 21.** 2 teaspoons = 1 dessert spoon.
2 dessert spoons = 1 table spoon.

So runs the usual comparison by chemists.

Assuming this to be accurate, and that there are 3 spoons of precisely the same pattern, what are their relative lengths, taking 100 for the largest ? Give results to the nearest whole number only.

*** 22.** Two parcels for the Parcels Post are precisely similar in shape and material, but not in size. They are wrapped up in brown paper and tied up with string. If their weights are 4 lb. and 5 lb. respectively, give the relative amounts of brown paper and string used, both as percentages of the larger to the nearest unit.

*** 23.** In the design of a ship, whose real length is to be 630 feet, a model on the scale $\frac{1}{4}$ in. to the foot is made. If her beam amidships is to be 70 feet, what should be the length and breadth of the model ? What is the ratio of comparative lengths in the real ship and her model ? If one of the decks of the ship has an area 34,560 sq. ft., what is (for that deck) the deck area of the model ? If the tonnage of the ship is 27,650, what is the tonnage of the model ?



*** 24.** Four weights, 1, 2, 4 and 7 lb., for Parcels Post are exactly similar. Taking unity as the edge of the base of the smallest, what are the corresponding measurements on the others ?

* 25. A hoarding has an advertisement on it, 10 ft. by 8 ft., showing a man holding in his hand an exactly similar paper 2 ft. 6 in. by 2 ft. On this there is again the hand holding the advertisement reduced in the same ratio as before. And so on *ad infinitum*. Show that the area of the fifth copy (sixth in all) is less than one millionth of the original.

* * 26. A set of three precisely similar spoons is made with relative capacities indicated in the Question **21**. They are electro-plate. Express the relative areas plated, taking 100 for the largest, and giving results only to the nearest whole number.

* * 27. Two bottles are precisely similar in *shape*, but not in *size*. [It is supposed that the thickness of the glass is proportional too.] The former is a "whole" bottle, and the latter is a "half" bottle. If the former is $11\frac{1}{2}$ inches high, how high is the latter? [Give the answer to the nearest eighth of an inch.]

* * 28. The same two bottles are labelled similarly. The area of the label on the "whole" bottle is 14 sq. in.; what should be the area of the label on the "half" bottle?

* * 29. The sizes of barrels of guns are designated according to the weight of a solid spherical lead ball which just fits them, and hence their diameters vary inversely as the cube roots of their numbers. In the case of a 12-bore, the ball fitting it weighs 12 to the pound, and measures 0·729 inch in diameter. Ounce bullets (No. 16) fit into a 16-bore gun of . . . . What is the diameter, in inches, of such a gun?

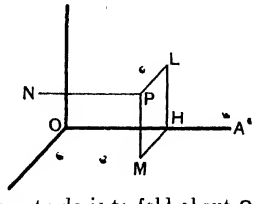
CHAPTER XLVIII.

PLANS AND ELEVATIONS.

§ 1. **Monge** (a French mathematician of about 1800) invented the method of representing *solid* bodies (such as houses, fortifications, etc.) by their plans and elevations, on *flat* paper, in such a way that accurate measurements could be made from them. Before his invention he had been received as a draughtsman and a pupil in the practical school attached to the military school at Mézières. The French Army authorities, at first, refused to receive his solution for the "défilement" of a proposed fortress, *because it had not taken so long as the official time allowed for the job* (which had hitherto involved long and tedious processes).

§ 2. The real essence of the problem is the representation, on the *flat*, of a point in *space*.

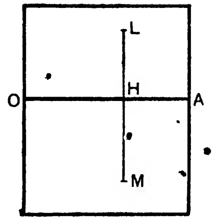
Now you will see, in the figure, that the position of the point P is **uniquely** determined, if we know HM, HL and HO.



Hence, all we have to do is to fold about OA and to represent HM, HL and HO on the flat. The position of P can then be recovered.

Figures, from which to work, and from which measurements can be made, are required by Engineers and Architects, who draw

Plans and Elevations. In the plan we have projections of points on to the Horizontal Plane (H.P.); in an elevation the projections of points on to a Vertical Plane (V.P.). In practice a front elevation, and often a side elevation, is used. The junction of the front vertical plane and the horizontal plane is often called the XY-line.



[In the figures, M is the "plan" of P, L is the "elevation" of P, OA is the XY-line.]

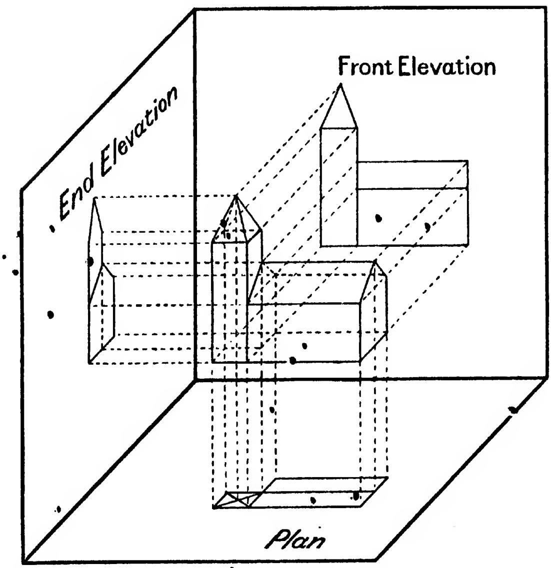
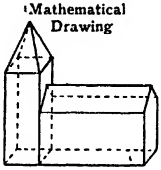


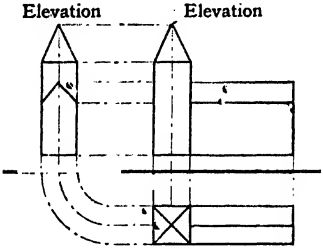
Figure to show how Plans and Elevations are conceived. With the exception of the Front Elevation these are *not* the actual figures to draw (the real plan etc. is not drawn sloping like this).

[You should study this figure very carefully and detect the position of a line missing in the End Elevation.]

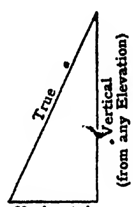
§ **3.** It is very tiresome to draw at all elaborate Plans and Elevations without a drawing board and T-square.

Below is an example of a **correct plan with front and end elevations.** *It is important to notice that, if we have any two of these, intersecting lines will give the third.* Being given a plan and elevation, we can get back to the original figure.





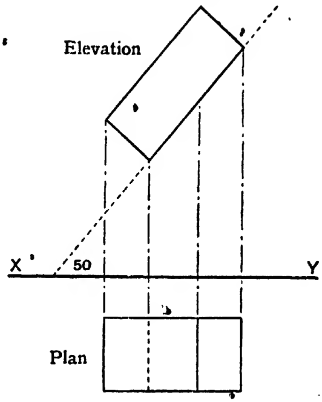
§ **4.** The **true length** of lines can be obtained from Plans and Elevations by putting the horizontal projection and vertical height at right angles. Very frequently it will be necessary to draw only one of these, using the other already drawn.



§ 5. The two **Examples** which follow show how tilting or twisting affects the problem.

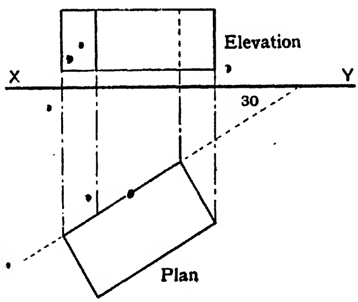
(1) Draw the plan and elevation of a brick, 9″ × 4½″ × 3½″, with its 4½″ edges perpendicular to the V.P. and its long edges inclined at an angle of 50° to the H.P.

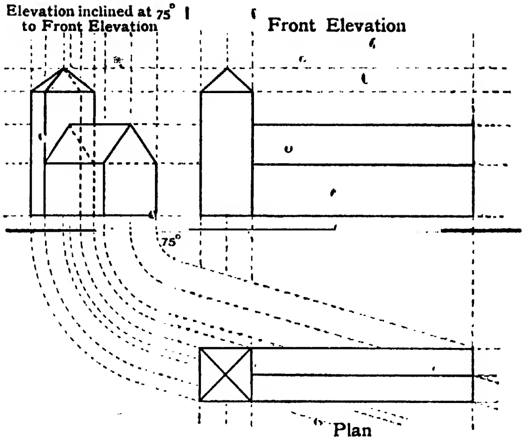
Since the inclination is to the H.P., draw the elevation on the V.P. first, making the 50° all right. Faint lines, perpendicular to XY, allow the plan to be drawn correctly. Only put in dotted lines that which would be below, and therefore invisible.



(2) Draw the plan and elevation of a brick, 9″ × 4½″ × 3½″, with its largest faces parallel to the H.P. and its longest edge inclined at an angle of 30° to the V.P.

Since the inclination is to the V.P., draw the plan on the H.P. first, making the 30° all right. Faint lines, perpendicular to XY, allow the elevation to be drawn correctly. Only put in dotted lines that which would be at the back, and so invisible.





§ 6. How to get elevations in other directions too is indicated in the figure above, where an **Elevation inclined at 75° to the front elevation** is shown. To avoid the confusion of a mass of lines, clearly the use of Indian ink for the essential things, together with the drastic use of India rubber, helps; but for examination purposes you need to show the construction lines faintly (*but clearly*), and then adopting different styles of dotting different types of lines (though tedious, if one is pressed for time) is not a bad plan.

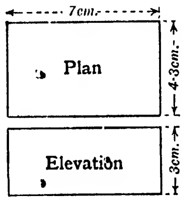
§ 7. The distance between two points, whose coordinates are a, b, c and p, q, r, is $\sqrt{(a-p)^2+(b-q)^2+(c-r)^2}$; as is easily seen if we consider the points as opposite corners of a right-parallelepiped. This fact is often useful in calculation.

EXAMPLES 48 (MOSTLY GEOMETRICAL DRAWING).

1. Three vertical posts, aA, bB, cC, of height 2, 4, 6 feet respectively, stand on horizontal ground, their lowest points A, B, C forming an equilateral triangle of side 6 feet. They support a flat board abc. To the scale of 1 cm.=1 foot, draw the plan ABC and the elevation of abc on the vertical plane through AB.

Obtain from your figure (showing your method) the real lengths of the sides of the board abc.

2. The figure shows the plan and elevation of a box. By drawing find the length of a diagonal of the box and its inclination to (1) the bottom, (2) the side, (3) the end, of the box.

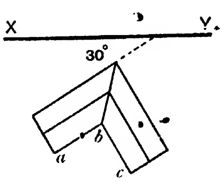


3. A plane slopes at 20° to the horizontal. A cubical tank, without a lid, edge 3 feet, rests on the plane, with an edge of the base horizontal. Draw, on a scale of an inch to the foot, a side elevation of the tank. Draw also a plan of the tank. The thickness of the walls of the tank may be disregarded.

4. The plan of a roof (without chimneys) of an L-shaped house is given. The centre line is the ridge.

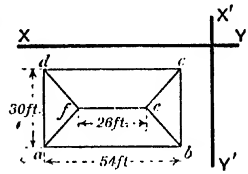
$$ab = bc = 40 \text{ ft.}$$

The width is 30 ft. The roof is 6 ft. high. Draw an elevation when inclined at 30° to the XY line as indicated.



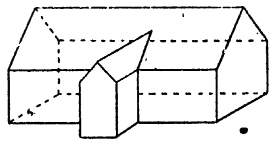
5. A prism is 3″ long. Its two ends are regular hexagons with sides 1″. It rests with one of its rectangular faces on the H.P. and with its axis inclined at 45° to the V.P. Draw its plan and elevation.

* 6. The figure shows the plan of the roof of a house, *ef* being the plan of the ridge and *a*, *b*, *c*, *d* the plans of the corners of the eaves. The ridge is 15 ft. vertically above the eaves. Draw the plan to scale, and then draw side and end elevations on vertical planes parallel to *cd* and *bc* respectively.

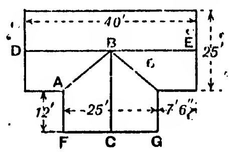


Determine, either by construction or by calculation, (1) the angles of inclination of the sides and ends of the roof to the ground, (2) the length of one of the four hip rafters (the plan of one is *af*) and its inclination to the ground.

* 7. Draw a plan, front and side elevations, of the accompanying building. You may choose any proportions you like.

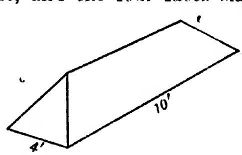


* 8. The figure represents the plan of a roof which consists of five surfaces, all inclined to the horizontal at an angle of 35°.



The surfaces all pass through B, the point where the ridge poles BC and DE meet. Draw the triangle CFG and determine the height of C above F or G. Then give a drawing of one of the surfaces which pass through the ridge pole BC, and determine (1) the true length of the valley rafter AB (2) the angle which this rafter makes with the horizontal.

* 9. A heap of stones by the roadside stands on a rectangular base measuring 10 feet by 4 feet, and the four faces slant upwards at an angle of 43° with the horizon, the figure being a *rough* sketch of the arrangement.

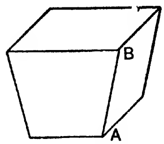


Draw a plan and elevation, the latter being projected on a vertical plane making angles 65° with the longer edges of the base.

Find the number of cubic feet in the heap.

* **10.** A table, 3 ft. by 2 ft., has legs at each of its four corners, and has a flap, 3 ft. by 1 ft. The table is 2 ft. 2 in. high. Draw its plan, with the flap opened, and with the 3 ft. length of the table inclined at 30° to the XY line. Draw its elevation too.

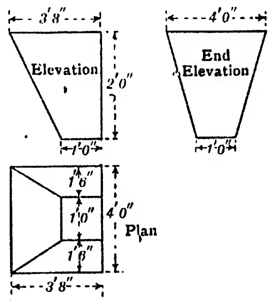
* **11.** A tub used for greenhouse plants is shown in a *rough* sketch in the figure. It measures 2 feet square at the base and 3 feet square at the top, and its height is 2 feet 6 inches. Draw a plan and elevation of the tub. On the same scale draw a figure of one of the slanting faces as it would look if laid flat on the plane of the paper. Find the slant-height (the distance between the parallel edges) of a slanting face, the length of an edge such as AB, the total surface area and the volume of the tub (outside measurement).



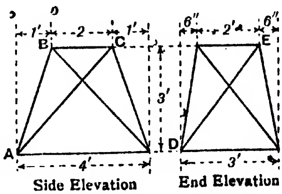
[The edges of the top and bottom squares are parallel. The line joining the centres of the squares is vertical.]

* **12.** The hopper shown by the *rough* diagrams has a closed bottom 1 foot square, and is open at the top. It is made of plate weighing 4 lb. per square foot.

Calculate the weight of the hopper.



* **13.** The side and end elevations of a trestle made of bars are shown in the figure. Find graphically the lengths of the bars AB, AC, DE. Measure and write down the lengths. Draw only (in clear, firm lines) as much of the trestle as may be required in order to obtain the solution. Scale one foot to an inch.



[*N.B.*—You should first sketch the plan. It must

consist of an outer rectangle ADFG (in which AD = FG = 4 ft and DF = GA = 3 ft.) and of an inner square PQRS centrally situated (with sides 2 ft., P being near A, Q near D, etc.), and also of twelve lines of which AS, AP and AQ are a sample. In the real trestle B is vertically above P, C above Q, also E above R. In each of the two elevations two identical things are shown, one exactly hiding the other.]

* **14.** Figures 1 and 2 show the end and side views of a shed. Draw the projection of the shed on a vertical plane equally inclined to the end and the side, measuring from the figure any dimensions you wish to transfer.

[Naturally you should draw this on an enlarged scale, and should state *clearly* (*certainly not on the figure itself*) in what ratio you increase lengths.]

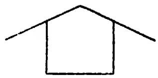


Fig. 1

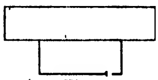
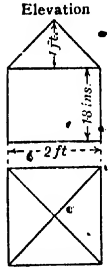


Fig. 2

* **15.** The figure shows the elevation and plan of a box with pointed roof containing an outdoor water meter. Draw a figure in ink, showing the appearance of the box when projected on a vertical plane making an angle of 30° with one of the vertical faces of the box. Also calculate the volume of the box.

[*N.B.*—The base is square.]

Elevation



Plan

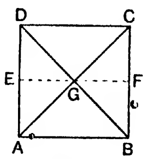


Fig. 1

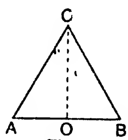


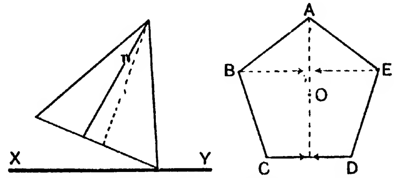
Fig. 2

* **16.** Fig. 1 shows the plan and Fig. 2 the elevation of a pyramid (forming the cap of a staircase banister post) on a square base ABCD, the elevation being shown in a plane parallel to AB. In Fig. 2 (that is, in elevation) ABC is an equilateral triangle. If AB $= 2a$, find the height of the pyramid and the length of a slant

edge. Find also its volume. Give the numerical coefficients to two significant figures.

Draw a figure showing the elevation projected on a vertical plane inclined to AB at an angle of 60°.

Find the area of the square section of the pyramid at a height y above the base.

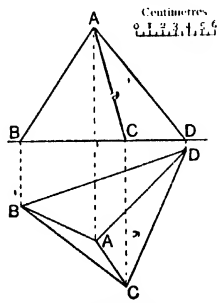


* **17.** The figure represents the elevation of a pyramid on a regular pentagonal base, the dotted line being the axis. The view is taken looking North. Draw the plan. Also draw (1) the elevation looking East, (2) the elevation looking West.

[HINT. You should first draw the correct base ABCDE.]

* **18.** Three equal hemispheres of diameter 10 cm. are placed symmetrically with their centres 12 cm. apart on a rough horizontal table; and a sphere of equal diameter is placed symmetrically so as to be supported by the hemispheres.

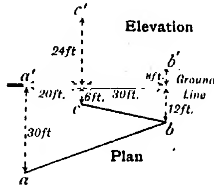
Draw a figure in plan and elevation.

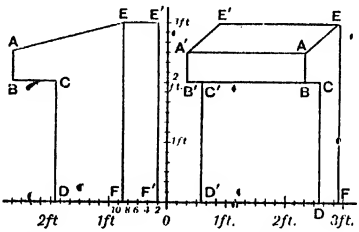


* **19.** The figure shows a triangular pyramid ABCD in plan and elevation. By pricking through, or otherwise, transfer the figure to your paper. Find the true length of AD. Draw the slant face ABD of the pyramid, and find the area of the face ABD in sq. cm.

* * **20.** A zig-zag path ABC runs in two straight portions AB and BC up the sloping face of a railway embankment, A being at the foot and C at the top.

The two portions of the path are shown in plan (ab, bc) and elevation ($a'b'$, $b'c'$) in the figure by the dimensions written on it, the figure not being drawn to scale. Draw a plan to scale, and on it draw the line in which the plane face of the embankment meets the horizontal plane; hence determine the angle which this plane face makes with the horizontal plane.

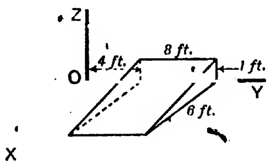




* * **21.** A writing desk is placed in an oblique position in a room, and its projections on two walls at right angles are shown in the figure. Draw figures in ink or *dark* pencil showing the desk in plan and its side and front elevations. Also calculate the area of the slant face of the desk and its inclination to the horizon.

* * **22.** A wedge-shaped frame, used for running cars down a step, is shown in the dimensioned sketch. On a suitable scale show its projection upon each co-ordinate plane.

Give the equation of the plane of its slant surface.

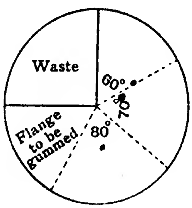


CHAPTER XLIX.

SOLID ANGLES.

§ 1. By this time we have got quite a clear idea about "solid angles" as the corners of solid figures; and the corners of furniture, familiar from our childhood, make us aware of them. The figures of the models of regular solids (on page 465) give us examples of various solid angles, and clearly some are greater than others.

Solid angles may be bounded by 3 or more (not less than 3) plane angles; and, to fix our ideas, a **solid angle**, bounded by the 3 plane angles 60°, 70° and 80°, **might be constructed** out of stiff paper. Draw a circle of *any* radius (about 2″ does well).



Mark off, in succession, the plane angles 60°, 70° and 80°. Allow for a considerable amount for joining (called the flange in the figure). Cut along the full lines, and bend along the dotted lines. Gum the flange and fasten it on to the adjacent angle. A good model is then made.

Solid angles bounded by 3 plane angles are called **trihedral angles.** If solid angles are bounded by more than 3 plane angles they are called **polyhedral angles.**

§ **2.** The question as to the **size of solid angles,** and the units in which they are measured, is considered next. In the same way that we gauge the size of a *plane* angle by the *length of the arc of a circle* of unit radius, with the angle at the centre, so do we measure the size of a *solid angle* by the *area* it subtends *on the surface of a sphere* of unit radius, whose centre is at the apex of the solid angle.

An attempt to construct (on the lines of § **1**) a solid angle, practically, with plane angles 100°, 51° and 47°, would soon convince us that necessarily **two angles of a trihedral angle must be greater than the third.*** It is equally clear that **the plane angles of a convex solid angle are less than 4 right angles**; for, in the circular figure (of § **1**), we cannot, at a point, make a total of more than 4 right angles. (The formal proof of this fact is given in Proposition **46**.) Absolutely 4 right angles makes the solid angle flat, indeed no angle at all, rather like the so-called straight angle in plane geometry.

If a solid angle is entirely on one side of each of its faces, it is said to be *convex*. It is as well to construct any *concave* solid angle, with plane angles (say) 70°, 60°, 50° and 40°, practically, [of course, an infinite number is possible.] The exercise gives a clearer idea of the distinction than any amount of reading. A solid angle is often understood to be convex, if the contrary is not specifically stated.

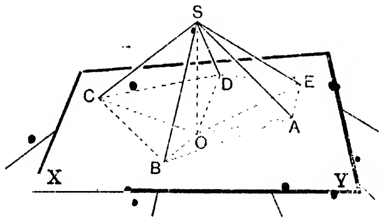
* Euclid's formal proof of this truth may be found in Appendix II., Proposition Z, on page 584.

PROPOSITION 46.

§ 3. **General Enunciation.** *The plane angles which form any (convex) solid angle are together less than four right angles.*

Particular Enunciation. The solid angle is at S, and the figure shows it as formed of five plane angles ASB, BSC, CSD, DSE and ESA. To prove that the sum of the plane angles at S must be less than four right angles.

Construction. Let any plane **XY** be drawn to cut all the arms of the solid angle on the same side forming the (convex) polygon ABCDE. In this polygon take any point O, join O to S and also O to the corners of the polygon.



Proof. Of the three plane angles at A,

$$\angle SAB + \angle SAE > \angle BAE,$$

$$i.e.\quad \angle SAB + \angle SAE > \angle OAB + \angle OAE.$$

Similarly at the points B, C, etc.

$\therefore$, by addition, the sum of all the base angles of the triangles whose vertices are at S is *greater* than the sum of all the base angles of the triangles whose vertices are at O.

Now there is the same number of triangles with vertex S as with vertex O.

But the sum of the angles of any plane triangle is two right angles, hence it follows that the sum of all the plane angles at S is *less* than the sum of all the plane angles at O; but this latter sum $=4$ rt. $\angle$s.

$\therefore$ the sum of the plane angles at S is less than 4 rt. $\angle$s.

Q.E.D.

EXAMPLES 49 (MOSTLY RIDERS).

O **1.** If the 3 angles of a triangle can be used to form a solid (trihedral) angle, of what shape must the triangle be ?

O **2.** All the sides of any convex plane polygon are produced in the same sense. Can all the exterior angles so formed be used to make a single solid angle ?

O **3.** Three angles of 120°, 130° and 140° are drawn separately on paper. They are cut out. Why would an attempt to make a single solid angle out of all three prove a failure ?

O **4.** Why must an attempt to make a solid angle from the three plane angles 27°, 65° and 93° prove a failure ?

O **5.** Why cannot the 3 angles of (1) a right-angled triangle or (2) an obtuse-angled triangle be used to make a solid (trihedral) angle practically ?

O **6.** Can a solid angle be made out of the 5 angles 60°, 70°, 80°, 90° and 100° (using all simultaneously) ?

7. The angles of a skew quadrilateral are together less than four right angles.

[*N.B.*—The four sides of a skew or gauche, quadrilateral are not in the same plane.]

8. OA, OB, OC are three straight lines drawn from a given point O not in the same plane, and OX is another straight line within the solid angle formed by OA, OB, OC : show that the sum of the angles AOX, BOX, COX is greater than half the sum of the angles BOC, COA, AOB.

9. With the notation of Question **8**, show that the sum of the angles AOX, COX is less than the sum of the angles AOB, BOC.

10. With the notation of the two preceding examples, show that the sum of the angles AOX, BOX, COX is less than the sum of the angles BOC, COA, AOB.

11. OA, OB, OC are three straight lines forming a solid angle at O, and OX bisects the plane angle AOB ; show that the angle XOC is less than half the sum of the angles BOC, COA. [X is on AB.]

12. A convex solid angle at O is contained by 4 faces. The plane angles are QOR, ROS, SOP and POQ. Prove that the sum of these 4 angles is less than twice the sum of the angles POR and QOS.

[HINT. Let the lines PR and SQ meet at X.

$\angle QOR < \angle ROX + \angle XOQ$, etc., etc., etc., and add.]

13. A solid angle at O is contained by 4 faces. The plane angles are BOC, COD, DOA and AOB. Prove that the sum of the 2 plane angles AOC and BOD is greater than the sum of the plane angles BOC and DOA. It is given that the solid angle is convex.

14. P is one corner of a regular octahedron and A, B, C and D are adjacent corners. If the diagonals of the octahedron meet at O, and if Q is *any* point on the circumcircle of ABCD, prove that the sum of the four angles BPC, CPD, DPA and APB is less than the sum of the four angles BQC, CQD, DQA and AQB.

****15.** L, M and N are the mid points of the three edges of a cube which pass through A. R is the corner of the cube diagonally opposite to A. How big is the solid angle at R, contained by the 3 plane angles MRN, NRL and LRM, compared with the whole possible solid angle at R?

[*N.B.*—You should use Spherical Trigonometry for this question.]

CHAPTER L.

EULER'S THEOREM.

§ 1. So far Three-dimensional Geometry has resolved itself into taking sections of solid figures and discovering their properties by applying to them the known principles of Two-dimensional Geometry, and in passing from section to section, or else in working with Plans and Elevations. In the theorem of this chapter ability to look upon the solid as a whole is necessary. Here models help very much.

Euler (1750), born at Basle in Switzerland, was one of the most distinguished mathematicians of the eighteenth century. Latterly he was blind, and his great genius was only surpassed by his greater industry, which did not cease because of his infirmity. He discovered the theorem which follows and which goes by his name.

PROPOSITION 47.

§ 2. **General Enunciation.** *In any polyhedron, bounded by plane faces,* $\mathbf{F} + \mathbf{V} = \mathbf{E} + 2$, *where* **F** *is the number of faces,* **V** *the number of vertices and* **E** *the number of edges.*

Proof. Imagine the polyhedron built up face by face.

When the *first* face is laid there are as many vertices as edges.

$$\therefore\ F_1 + V_1 = E_1 + 1.$$

When a *second* face is laid (having one edge in common with the first), there is one less fresh vertex than fresh edge.

$$\therefore\ F_2 + V_2 = E_2 + 1.$$

When a *third* face is laid, there is again one less fresh vertex than fresh edge.

$$\therefore\ F_3 + V_3 = E_3 + 1,$$

and so on, until the last but one case.

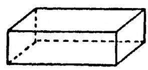
$$\therefore\ F_{n-1} + V_{n-1} = E_{n-1} + 1.$$

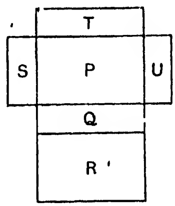
Now when the (single) last face is laid, to complete the polyhedron, no fresh vertices nor edges are made, so in this case

$$F + V = E + 2.$$

Q.E.D.

§ 3. The proof of this proposition assumes that the laying is carried on, avoiding leaving (temporarily) any polygonal aperture, to be closed by adding another face just before the end. If these openings are left, the argument is not sound, though the final conclusion is correct. For instance, this brick-shaped polyhedron might be imagined built up in several ways. In some the preceding argument is unsound.





Argument sound.

(1) Floor P.
(2) Front wall Q.
(3) Ceiling R.
(4) Left end S.
(5) Back wall T.
(6) Right end U.

Argument unsound.

Floor P.
Front wall Q.
Ceiling R.
Back wall T.
Left end S.
Right end U.

The argument is unsound at (4), for then two openings would be left, like the cover of a match-box.

Steps.	Faces in position.	F	V	F+V	E	Conclusion.
1	P	1	4	5	4	$F+V=E+1$
2	PQ	2	6	8	7	same
3	PQR	3	8	11	10	same
4	PQRS	4	8	12	11	same
5	PQRST	5	8	13	12	same
6	PQRSTU	6	8	14	12	$F+V=E+2$

Perhaps you will not, at once, appreciate the truth of this proposition only by reading the proof. You are advised to consider building up a brick-shaped solid and to see that the numbers

for F, V and E are correct at each step, and then to try the same for any other shaped solid with plane faces, such as a greenhouse, pyramid, etc., etc.; but the point is, try it for some shape with which you are quite familiar. (Don't forget to reckon the floor as one of the faces.)

EXAMPLES 50 (GENERAL CASES).

There are no Answers given at the end of the book.

1. What is the number of edges in the following Prisms ?

(1) With triangular base.
(2) With quadrilateral base.
(3) With pentagonal base.
(4) With hexagonal base.
(5) With n-agonal base.

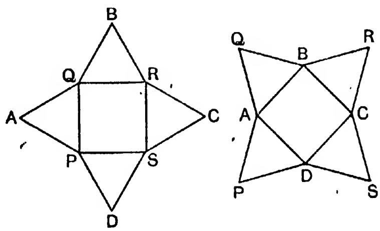
If "Pyramids" is substituted for "Prisms" in the preceding, what are the 5 answers ?

2. Fill up the following table. You are advised to fill in the numbers in pencil first, and not to ink them in before you are quite sure they are right. References to various pages where the figures are described, and a figure given, are made. The relation $F+V=E+2$ should be satisfied in every case.

No.	Solid.	Figure on Page.	F	V	F+V	E
1	Parallelepiped - -	423				
2	Tetrahedron - -	423				
3	Cube - - - -	465				
4	Octahedron - -	465				
5	Solid, like tub for greenhouse plants -	547				
6	Heap of stones - -	546				

* **3.** Read the heading for Question **2**, and fill up the following table. [Don't forget to count the floor as one of the faces.]

No.	Solid.	Figure on page.	F	V	F+V	E
1	Water meter box - -	548				
2	Skylight - - -	449				
3	Roof of house - -	546				
4	Building - - -	546				



* **4.** Two equal pieces of paper, of the form shown, are in the shape of equal squares with equilateral triangles externally on the sides. The two pieces of paper are fitted together, with the triangles bent, to form one solid figure. [The point A goes into the re-entrant A, etc., the 2 squares form the top and bottom, and 8 triangles the sides of the solid.] How many faces, vertices, and edges has it? Is the relation $F+V=E+2$ correctly satisfied?

* * **5.** Read the heading for Question **2** and fill up the following:

No.	Solid.	Figure on page.	F	V	F+V	E
1	Rhombic dodecahedron	470				
2	Church - - -	542				
3	Truncated cube -	514				
4	Cubo-octahedron -	515				

** 6. The mid points of neighbouring edges of a regular octahedron are joined to form a solid. Two opposite faces are squares; and these are connected by 16 triangles. What are F, V and E for this solid ? Is the relation $F+V=E+2$ satisfied ? Show that 8 of the triangles are equilateral, and that the other 8 are right-angled.

** 7. Prove that the sum of all the plane angles of the faces of any solid (bounded by plane faces) is $4(V-2)$ right angles, where V is the number of its vertices.

CHAPTER LI.

THE EARTH.

§ 1. **Geometry dates** from very many years before Christ, and its cradle was, possibly, in Egypt. Perhaps it was designed primarily (for revenue purposes) for surveying lands likely to be inundated by Nile floods. As the knowledge of the **Earth** grew, it came to have a wider significance; but the very name (γῆ μέτρον) shows that measurement of the Earth was one of its objects. The science has now a much larger meaning; however, its great importance in matters connected with the Earth, and beyond too, cannot be forgotten.

§ 2. **Many of the heavenly bodies are nearly spherical**; and, in particular, the Earth, with which we are most intimately concerned, affords many problems of vital importance to us. In the chapter on Spheres we have learnt something about spheres in general; and it has frequently been assumed, in earlier portions of the book, that plane sections of spheres are circular, so that (in considering circles) we have been accustomed to deal with "the distance of the visible horizon" and "the mutual visibility of points" on the surface of the Earth [Vol. II. p. 296, p. 396, etc.]. In this chapter we shall study the problem in greater detail.

§ 3. We have already considered **great circles** of a sphere. These are of enormous importance on the Earth, especially in respect to Navigation. Meridians of longitude are all "great circles"; of the parallels of latitude, the Equator only is a "great circle."

§ 4. The **shortest line** (on the surface of a globe) between two points is the portion of an arc of a circle, whose plane passes through both points and the centre of the Earth. It is thus an arc of a "great" circle. When the points are fairly close to the Equator, there is only a slight gain in distance (for courses on the surface of the globe) in travelling along the "great" circle, if the difference of longitude is not large; but absolutely the shortest

way between two places, both in the same latitude, is *not*, in general, along the parallel of latitude. However, when the places are in high latitudes (with a considerable difference of longitude too), the gain of distance in travelling by a " great-circle-route " is very appreciable [this is entirely distinct from any departure from an exactly spherical shape of the Earth].

To make it quite clear, take 2 points on the surface of a globe (essentially a globe and not a map out of an atlas), and stretch a string *tightly* on the surface of the globe between them. The string, being tight, will give the shortest track. In this connection you should study the figure on page 565, also reading the verbal explanation.

* § **5.** In **Navigation** it is shortest to go by " great " circles, and " great-circle-sailing " is the quickest; but consideration of the intervention of land, currents, wind, ice, unduly high latitudes, or other difficulties, generally makes a modification desirable. In practice ships rarely sail absolutely on great circles, but on a series of lines, which (though straight on the chart) nearly represent the curves, so that the change of course is in jumps rather than continuous. The difference in length between these straight pieces and the absolute great circle is negligible. The establishment of " steam lanes," where there is much ocean traffic, minimizes the risk of collision, for the outward and homeward bound traffic keep to different lanes; also, in the event of any particular vessel being in distress, there is a good chance of finding some other vessel in the vicinity to render assistance. " Lanes " were instituted long before the days of wireless telegraphy. But, even in these days, the " great " circle will be always followed, when possible, by practical seamen. Time is precious, and every mile saved is a saving in other ways too numerous to mention.

For Navigation by Air, where there is no question of the intervention of land, the absolute necessity of aerodromes and friendly inhabitants (in case of an enforced landing), and other considerations too, generally makes the shortest course perhaps not the wisest.

§ 6. *Since a sphere is not developable, it is impossible to represent on paper (which can be flattened out) a portion of the surface of the Earth absolutely correctly.* Yet, perforce, we need to use maps printed on flat paper, for globes alone would be far too bulky and costly. The curvature of the surface of the Earth is trivial (as far as a map is concerned) in a small area, and then it does perfectly well to assume the surface of the Earth (or rather the sea produced in all directions) to be quite flat, but when larger areas, such as countries, continents and even hemispheres (or more) are under consideration, the problem of how best to show a *spherical* surface on a *plane* map remains.

A map **Projection** is the system on which parallels of latitude and meridians of longitude are represented on paper. In any particular case some feature, such as correct areas, no local distortion, correct directions, etc., is often especially important. There are many projections from which to choose, some lay stress on one feature and some on another. Others adopt a sort of golden mean, in them nothing is very good and nothing is very bad.

In this book there are ILLUSTRATIONS of (1) a globe, and only two (of the many) projections, viz. (2) Mercator's Projection and (3) the Central, or Gnomonic, Projection.

" Shortest courses " (necessarily on " great circles "), " rhumb lines " (lines of constant true bearing or loxodromes, as they are called), and " *equal* circular areas " (circular islands which on the Earth are about the size of Spain and Portugal) are shown. Their study brings out the very great difference.

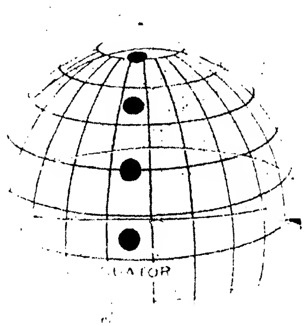
The methods of the construction of various " projections " are too hard for an elementary text-book ; but everyone has to use maps of the world, and it is as well to know their limitations.

§ 7. **A description of the Illustration.** First, 3 pieces of string were stretched tight on the surface of the globe between certain places. Their ends were fixed with plasticine. *The strings, being tight, must show arcs of "great circles."*

(1) The southernmost is from 10° N. lat. to 10° N. lat., with a range of 90° (or 6 hours). It is noticeable that the "great circle" is not very far from the parallel of latitude.

(2) The middle is again from 10° N. lat. to 10° N. lat., but the range (10 hours) is much greater. It goes considerably further north. The greater range makes a lot of difference.

(3) The northernmost is from 45° N. lat. to 60° N. lat., with a range of 10 hours. Now 45° N. lat. is *much* south of British latitudes, and yet the "great" circle approaches the Pole.

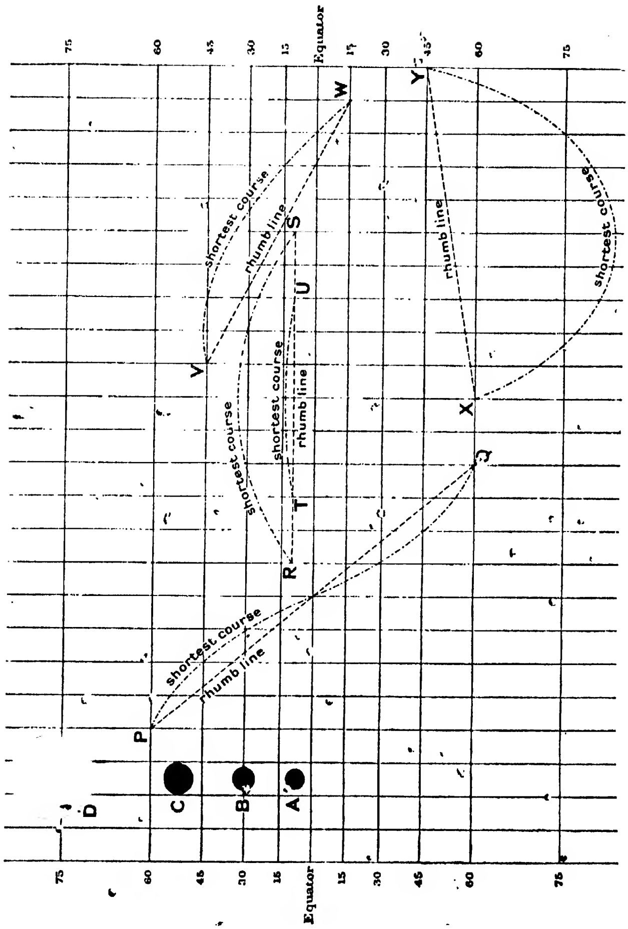


Secondly, 4 circular islands (of about 280 miles radius, about the size of Spain and Portugal) were fixed on the globe, with their centres at about $6\frac{1}{2}$° N. lat., 30° N. lat., $52\frac{1}{2}$° N. lat., 76° N. lat.

The circles were all the same size, and therefore represent *Equal Areas*. Notice (4) the southernmost is far from the meridians on each side.

(5) The next is rather nearer, but not very much.

(6) The $52\frac{1}{2}$° circle gets quite close to the adjacent meridians, while (7) the 76° circle overlaps the meridians very much.


Equator
75
60
45
30
15
shortest course
rhumb line
A
B
C
D
P
Q
R
S
T
U
V
W
X
Y

§8. MERCATOR'S PROJECTION.

Distances. The scale increases as you go away from the Equator. Note that from 60° N. lat. to 75° N. lat. (a range of 15° of latitude) is nearly 3 times as long as from 0° to 15°.

Degrees of latitude on the globe are equal, but different on the projection; on the other hand, the reverse is true for degrees of longitude.

Areas. *Small* areas preserve their shape but not their size. Note that on the globe A, B, C and D are *equal* circles, so that their diameters are of the same length on the globe. This is far from the case on this projection.

N.B.—Near the equator it is hard to detect that A and B are not exact circles, though B is distinctly larger than A. C (about the latitude of England) has somewhat of a bulge towards the top, while the size of D (about the latitude of Spitsbergen) makes it ludicrous.

Directions. LINES OF CONSTANT TRUE BEARING (Rhumb Lines or Loxodromes as they are called) ARE STRAIGHT. This is the feature of the Projection. It makes its use universal at sea. Admiralty Charts are on this Projection. Shortest courses ("great" circles) are in general curved.

N.B.—Study the shortest courses and rhumb lines PQ, RS, TU, VW and XY.

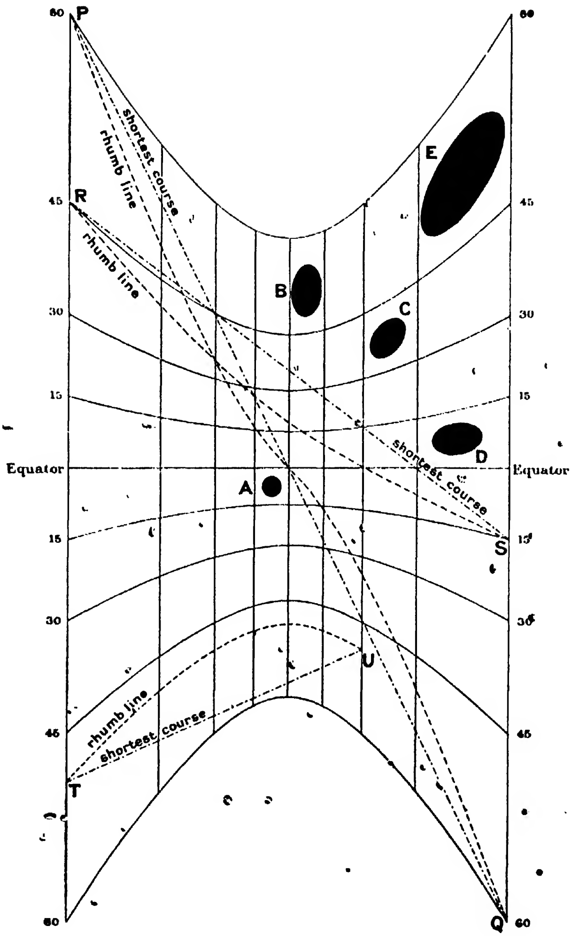
The curve RS is far more pronounced than TU. They both start and end in 10° N. lat., but the range of the former (10 hours or 150°) is much wider than the range of the latter (6 hours or 90°).

To go from V to W (and W is *much South* of V) notice that, for the shortest route, you go distinctly *North* to start with.

In travelling from X to Y (and Y is well *North* of X) *much* of the shortest course is *South*. Remember that the scale varies, and you cannot trust too readily to appearances on this Projection for distance, but on the globe you could see that about $\frac{4}{5}$ of the course is South of the starting point.

The projection is **not perspective.** There is no geometrical way of imagining it.

Its **Inventor** was **Gerhard Kremer**, about 1550. (*Krämer* is the German for "Trader," and the name of the inventor was latinized to Gerardus Mercator.)


60
P
60
shortest course
rhumb line
E
45
R
45
rhumb line
B
C
30
30
15
15
shortest course
D
Equator
Equator
A
15
S
15
30
30
U
rhumb line
shortest course
45
45
T
60
Q
60

§ 9. CENTRAL OR GNOMONIC PROJECTION.

Distances. Variable. Scales may be quite different, even in places of the same latitude, and are different in the same place in different directions.

Areas. Distorted in size and shape.

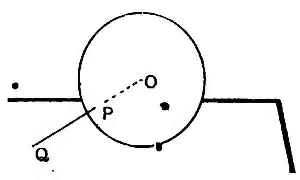
Directions. SHORTEST COURSES (*i.e.* "great" circles) ARE STRAIGHT LINES. This is the feature of the projection, which can be used for the study of direct routes. Lines of constant true-bearing (rhumb lines) are, in general, curves.

It is **Perspective.** One can imagine one's eyes at the centre of the globe, and a straight line from it to any spot on the globe cutting the plane of projection. Great circles on the globe thus become straight lines on the plane.

Another way is to imagine a hollow wire cage, showing the parallels and meridians, with a point of light at the centre of the globe. The shadows cast by the wires, on a flat piece of paper, are on this projection.

The diagram shows how points (like P) on the surface of a sphere are represented by points (like Q) on a plane.

The name "**gnomonic**" was given because the problem of making such a projection was like the problem of the construction of a sun-dial.



N.B.—Study the shortest courses and rhumb lines. The ludicrous distortion of E (recollect that on the globe E is the same size and shape as A) makes it quite unsuitable for most purposes.

One object of illustrating it is to answer the question: "Well, as shortest courses on Mercator's Projection *are not* straight lines, why not make a Projection on which shortest courses *are* straight lines?" It certainly affords a striking example of the practical difficulties of Solid Geometry.

§ **10.** Very nearly all problems on "great-circle-sailing" can be done by **plane trigonometry** (of course indirectly) if either both the latitudes, or both the longitudes, of start and finish are the same; but by **spherical trigonometry** preferably in other cases.

[Spherical Trigonometry is like Plane Trigonometry. The latter is only a special case of the former when the radius of the sphere is infinite. Spherical Trigonometry is specially adapted to problems on the surface of the Earth (which is so nearly spherical), and for problems concerned with stars, etc., on the Celestial Sphere.]

§ **11. To sum up the lessons that can be learnt from the study of the globe and these two projections.**

(1) Globes can be correct, but are far too bulky and costly for most purposes.

(2) Flat maps of spheres cannot be perfect.

(3) We can choose to represent one feature correctly at the expense of others, but with too large an area the distortion becomes irritatingly big.

EXAMPLES 51 (CALCULATIONS. SPECIAL CASES).

N.B.—The "distance of the visible horizon" and the "mutual visibility of points" has been considered on pages 296, 396, etc., Vol. II., and exercises are proposed in the examples following those pages. Comparatively few will be found here.

1. If the Earth is a sphere of about 4000 miles radius, and the population of the world is estimated at 2000 millions, how many individuals to the square mile (sea and land) are there on the average?

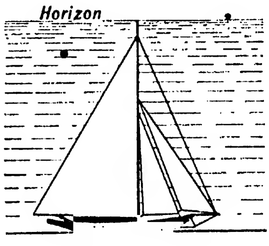
[Clearly only the roughest calculations are appropriate to such data. Only give the result to one significant figure.]

2. Suppose that water, to the average depth of 1 mile, covered the whole surface of the Earth (a sphere of 3960 miles radius).

If all that water were concentrated in a single sphere, what would be its diameter? Give the result correct to two significant figures.

If the depth had been $\frac{1}{2}$ a mile, but the data unaltered otherwise, what should be the answer?

3. I am in a window overlooking the sea, and observe a yacht with which I am well acquainted. The top of her mast appears to be in the horizon beyond. [The figure, which of course is not drawn to scale, conveys the idea.] My eyes are at a height of 50 feet above the sea, and the top of her mast is 40 feet above the water. How many yards is the yacht away?



[One difficulty in practice is to have such accurate data; a small discrepancy in either may mean so much. It is more sensible to give an answer in round numbers.]

If the heights were respectively 54 feet and 58 feet, what should be the answer?

* **4.** Suppose that an aeroplane was up as much as 35,000 ft. above Shrewsbury and that the furthest points in England are 225 miles from that place [Shrewsbury is more or less central as far as England is concerned]. Is the whole of England visible to the aviator (quite apart from the strength of his eyesight, etc.)?

* **5.** If h be the height of an aeroplane and R the radius of the earth (*both expressed in the same units*), the area of the Earth visible is $\dfrac{2\pi R^2 h}{R+h}$ (expressed in squares of the same units).

In particular, how many *square miles* are visible from a height of 4000 *feet*? [The Earth is a sphere of 3960 miles radius.] Give the result correct to one significant figure (and no further).

* **6.** The Arctic Circle is in North latitude $66\frac{1}{2}°$. What area is included therein right up to the North Pole? [Radius of the Earth 3960 miles.]

* **7.** Find the area on the surface of the Earth included between the parallels of latitude 50° N. and 55° N., taking the radius of the Earth to be 4000 miles.

**** 8.** Two places, A and B, are both situated in North latitude 70° and their longitudes differ by 40°. It is required to find the difference in distance between A and B, according as (i) a route along the parallel of latitude or (ii) a great circle route is adopted. The circumference of the Earth is 21,600 nautical miles.

[HINT. (i) Calculate the length of the arc AB of the circle through the parallel of latitude. (ii) In the same calculate the chord AB. (iii) If O is the centre of the globe, use the $\triangle$AOB and calculate the great circle arc AB.]

*** * 9.** Berwick-on-Tweed, 56° N. lat., 2° W. long. Moscow, 56° N. lat., 38° E. long. What is the distance between these two places (1) by parallel of latitude (or rhumb line), (2) direct, by a straight line through the Earth (*i.e.* a chord of the sphere), and (3) by the great circle route? Take the radius of the Earth to be 3960 miles, and use 4-figure logarithms. Show that there is a saving of not much over 20 miles if (3), rather than (1), is adopted.

*** * 10.** The problem is to sail by the shortest course from the south of Africa to the south-western corner of Australia. Assuming that the latitude of start and finish is 35° S., and that the respective longitudes are 20° E. and 115° E., how many miles is saved by adopting the great circle course rather than the parallel of latitude? What is the furthest south reached by the great circle?

[The Earth is a sphere of 21,600 nautical miles circumference.]

*** * 11.** An aeroplane is to fly from South Ireland (say about $51\frac{1}{2}$° N. lat., 10° W. long.) to Newfoundland (say about $51\frac{1}{2}$° N. lat., 55° W. long.).

The Earth is a sphere of 21,600 nautical miles circumference.

Calculate (1) the distance along the parallel of latitude, (2) the length of the chord between those two spots, (3) what angle that chord subtends at the centre of the Earth, and (4) the length of the "great circle course" between those two spots. (5) About how much distance is saved by adopting the "great circle" instead of the parallel of latitude? (6) What is the latitude of the furthest North touched?

** 12. Bombay - - 19° N. lat., 73° E. long.,
Mexico City - 19° N. lat., 99° W. long.

An aeroplane has to fly, by the shortest route, from one of these places to the other. What is the highest latitude reached ? (Certainly, use Spherical Trigonometry and, preferably, 7-figure tables.)

[*Note.* Though both start and finish are in the *Tropics*, there is such a very wide difference of longitude between them, that the shortest course comes even nearer the N. Pole than the *Arctic Circle* (it crosses over Spitsbergen) ; as a matter of fact the great-circle-course is nearly 1500 miles shorter than the parallel-of-latitude-course.]

APPENDIX I.

ON METHODS OF SHOWING RELIEF.

In this Appendix is a note on some of the practical ways adopted for showing relief.

§ 1. **Models** to scale of course convey most readily to the eye correct ideas of **Relief**; but their use in the field, to show features of the ground, is entirely out of the question owing to their cost, weight and bulk. Indoors, "relief maps" (usually on a small scale) are extremely useful, though all the ups and downs have to be enormously exaggerated to make them apparent at all. [If a model, the same size as a billiard ball, were made of the Earth, the height of Mount Everest (if unexaggerated) would be less than half the thickness of one page of this book.] Models (*papier maché*, tin, wood and plasticine, sand-tables) are impracticable *out of doors*, but they are extremely valuable *indoors*.

§ 2. *The problem of how best, on flat paper, to show relief remains.* There are a variety of systems :

(1) **Hachures.**
(2) **Hill-shading.**
(3) **Contours** (including **Layers**).

Also **Colours,** in any of these cases, frequently help to convey meanings quickly to the eye.

Combinations of the various systems are generally used.

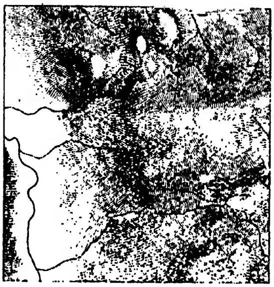
Some advantages and disadvantages in each are here considered.

(1) **Hachures** are fine lines always going down the *lines of greatest slope* (that is, the way water unimpeded would flow down). Nicely drawn, they show hill features well; but they have to be drawn close together for steep slopes, and their very closeness obscures detail.

(2) **Hill-shading** conveys good impressions, and so is useful in the general study of a map; but is not exact enough for any detailed measurements. Both oblique and vertical illumination are employed. The former gives different shading to slopes of equal steepness, according to their aspect to the light, and so may be misleading. The light comes from the top left-hand corner. In the latter the light is right overhead. Flat surfaces (whether on high ground or low ground) get the full light, and remain unshaded on the map. Slopes get darker and darker as the steepness increases. The latter (vertical illumination) only is illustrated here.

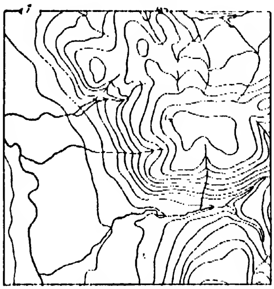
(3) **Contours** are fine lines, every point on each of which is at the same height above the datum level (usually sea-level). They are of immense value out of doors. More or less accurate calculations can be based on measurements made from a map on which contours are drawn. The meaning of contours can be readily understood by pouring water into a bowl containing a model. The water-line, on the model, at different depths will show the contours. [Contours are at right angles to hachures.] By themselves contours do not give to the eye a good idea of relief.

1. Hachures.

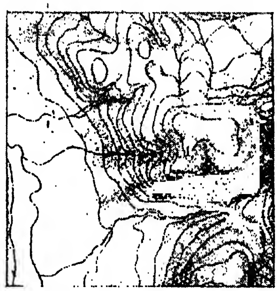


2. Hill-shading.

3. Contours.



4. (Nos. 2 and 3 in combination.)



§ 3. The figures give examples of the *same* area, showing the advantages and disadvantages. Naturally with colour there is never any question as to whether a line is a contour or a river; but of course it is easy to settle this point, by map-reading, almost at a glance.

Generally, the question comes, "Well, which is best?" No definite answer can be given. It all depends upon the use to which the map will be put, and upon the time and labour that can be bestowed on its preparation.

§ **4.** When bands of colour are used on Maps, to cover the ground between contours, we get the **Layer System.** This is useful for giving a good general impression of the relief of a *large area*, and in consequence the system is most valuable in small scale maps. Whenever the contour lines are drawn close together, it would naturally be impracticable to find a different colour for each interval. Each different shade may therefore include several such intervals. To give the impression of rising ground it is necessary to use colours which merge into one another, otherwise a stepped appearance will result.* Following the colours of the Spectrum, Blue (sea), Green, Yellow, Orange, Red (highest), with intermediate tints greenish-yellow, etc., gives a good appearance. It is expensive in printing, for each map must go through the press several times, and of course the "registration" must be perfect (*i.e.* successive printings must not show any overlappings or gaps).

§ **5.** Naturally any system is practically useless for a map of an area which has little variation in height.

§ **6.** There is another way of showing heights accurately. This is by **Bench Marks** and **Spot Heights** (usually called **Spot Levels**). Against any particular place is given a number telling the height. The impossibility of visualizing relief from these numbers perhaps makes their mention inappropriate here; but they afford a basis for calculations more accurate than from heights obtained from contours. [Bench Marks are like ↑. They are to be seen throughout the country. Their heights above sea-level are accurately known and are marked on the British 6″ to 1 mile maps.]

* See the figure on page 498.

APPENDIX II.

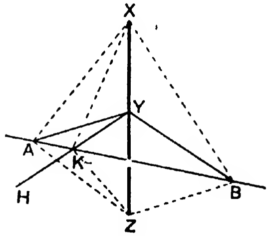
PROOFS OF FIVE PROPOSITIONS ASSUMED BY SOME.

In this Appendix are given formal proofs of certain Propositions, the truth of which some take for granted.

PROPOSITION V.

General Enunciation. *If a straight line is perpendicular to each of two straight lines at their point of intersection, it is perpendicular to every straight line passing through this point of intersection and lying in the plane containing them (and hence is perpendicular to the plane that they determine).*

Particular Enunciation. XY is perpendicular to YA and YB. To prove that XY is perpendicular to any line, through Y, in the plane determined by YA and YB, and hence to that plane itself.



Construction. Produce XY to Z, doubling it. Let YKH be *any* line through Y in the plane of YA and YB. AKB is *any* straight line, in that plane, cutting these lines at K, A and B. Join XA, XK, XB, ZA, ZK and ZB.

Proof. YA perpendicularly bisects XZ. $\therefore$ XA $=$ ZA.

Similarly XB $=$ ZB,

so that $\triangle$s XBA, ZBA are congruent (three sides).

$\therefore$, in particular, $\angle$XBA $=$ $\angle$ZBA ;

hence $\triangle$s XBK, ZBK are congruent (two sides and included angle).

$\therefore$, in particular, XK $=$ ZK.

Now $\triangle$s XYK, ZYK are congruent (three sides) ;

$\therefore$, in particular, $\angle$XYK $=$ $\angle$ZYK,

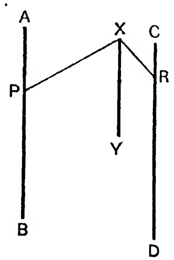
and these are adjacent angles and equal, and hence right angles, so that XY is perpendicular to *any* line, through Y, in the plane of YA and YB.

Hence, also, XY is perpendicular to the plane determined by YA and YB. **Q.E.D.**

PROPOSITION W.

General Enunciation. *If two straight lines are parallel to a third (not in their plane), they are parallel to one another.*

Particular Enunciation. AB and CD are both parallel to XY (not in their plane). To prove AB and CD parallel.



Construction. In the plane of XY and AB draw XP, perpendicular to XY, to meet AB at P; likewise, in the plane of XY and CD draw XR, perpendicular to XY, to meet CD at R.

Proof. XY is at right-angles to two lines, XP and XR, at their point of intersection X, and hence XY is perpendicular to the plane **PXR**.

Now AB and CD are parallel to XY, and hence are both perpendicular to this same plane.

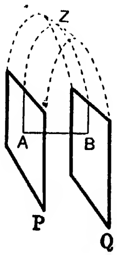
∴ AB and CD are parallel.

Q.E.D.

PROPOSITION X.

General Enunciation. *Planes to which the same straight line is perpendicular are parallel to one another.*

Particular Enunciation. AB is perpendicular to each of the planes **P** and **Q**. To prove those two planes parallel.



Construction. If the planes are not parallel, let them meet, and let Z be one of the points on their junction. Join ZA and ZB (necessarily in the planes **P** and **Q** respectively).

Proof. ZAB is a triangle, and *two* of its angles, A and B, are each right angles (for, when a line is perpendicular to a plane, it is at right angles to any line in the plane),
and this is impossible;

∴ the planes **P** and **Q** cannot meet;

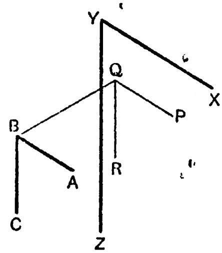
in other words the planes **P** and **Q** are parallel.

Q.E.D.

PROPOSITION Y.

General Enunciation. *If a pair of intersecting straight lines are respectively parallel to another pair of intersecting straight lines, the plane containing the first pair is parallel to the plane containing the second pair.*

Particular Enunciation. AB is parallel to XY and CB is parallel to ZY. To prove that the planes **ABC** and **XYZ** are parallel.



Construction. Draw BQ perpendicular to the plane **XYZ.** Through Q draw PQ and RQ parallel to XY and ZY (necessarily in the plane **XYZ**).

Proof. AB and PQ are both parallel to XY ;

$\therefore$ AB and PQ are parallel ;

$\therefore$ $\angle$ABQ + $\angle$BQP = 2 rt. $\angle$s ;

but $\angle$BQP = 1 rt. $\angle$ (construction) ;

$\therefore$ $\angle$ABQ = 1 rt. $\angle$.

Similarly $\angle$CBQ = 1 rt. $\angle$.

Hence BQ is perpendicular to *two intersecting lines* AB and BC, at their common point B.

∴ BQ is perpendicular to the plane **ABC**; but BQ is, by construction, perpendicular to the plane **XYZ**, *so that the same straight line* BQ *is perpendicular to the two planes* **ABC** *and* **XYZ**,

∴ the planes **ABC** and **XYZ** are parallel (using the preceding proposition).

Q.E.D.

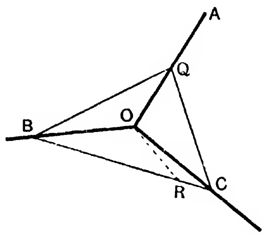
PROPOSITION Z.

General Enunciation. *If a solid angle is contained by three plane angles (i.e. a trihedral angle), any two of these are together greater than the third.*

Particular Enunciation. The solid angle is at O. The three plane angles are BOC, COA and AOB

[If one of these (viz. $\angle$BOC) is equal to, or less than, either of the other two (viz. $\angle$BOA or $\angle$AOC), there is no difficulty in establishing the truth that $\angle$BOA + $\angle$AOC > $\angle$BOC; and so we shall only consider the case of $\angle$BOC being greater than either $\angle$BOA or $\angle$AOC.]

To prove $\angle$BOA + $\angle$AOC > $\angle$BOC.



Construction. In OB and OC take any points B and C. Join BC. In the plane **BOC** make the angle BOR equal to BOA, the point R being on BC. From OA cut off OQ equal to OR. Join BQ and QC.

Proof. In the $\triangle$s BOQ and BOR,

$$\because \begin{cases} \text{BO common,} \\ \text{OQ} = \text{OR (construction),} \\ \angle\text{BOQ} = \angle\text{BOR (construction);} \end{cases}$$

$\therefore$ $\triangle$BOQ = $\triangle$BOR, and, in particular, BQ = BR.

Now, in the triangle BQC, two sides are greater than the third ;

$$\therefore\ BQ + QC > BC\ ;$$

$$\text{but}\quad BQ = BR\ \text{(proved)}\ ;$$

$$\therefore\ QC > RC.$$

Again, in the $\triangle$s QOC and ROC,

$$\because\ \begin{cases} QO = RO\ \text{(construction)}, \\ OC\ \text{common}\ ; \\ \text{but}\ QC > RC\ ; \end{cases}$$

$$\therefore\ \angle QOC > \angle ROC.*$$

To each add the equal angles BOQ and BOR ;

$$\therefore\ \angle BOQ + \angle QOC > \angle BOC\ ;$$

$$\therefore\ \angle BOA + \angle AOC > \angle BOC.$$

Q.E.D.

* If the truth of this is not assumed, reference should be made to Euclid I. 25.

ANSWERS.

NOTE AS TO ANSWERS.

There are often many correct forms in which an answer may be given, and it is quite possible that very little arithmetic is required to change one into the other.

In the following are answers to three types of questions:

(1) *G.D.*, many of which are intended to be solved by G.D. alone (these are confined to some of the questions on "Plans and Elevations" in Examples **48**, but by no means on all); for them lengths have been given approximately and angles to half-degrees.

(2) *Calculations, Special Cases,* which can be solved either by G.D. alone or by calculation alone (preferably by both, using the former as a check to the latter. *N.B.*—It is generally wiser to make the rough check *before* the more accurate work). Clearly more exact results should be obtained by the second method. In the following, frequently, only the answer obtained by the use of tables is given (even then perhaps the data do not warrant an excessive number of figures).

(3) *Calculations, General Cases* [where surds come in to the answer they have not been evaluated (nor, in general, should they be); but numerical denominators have been rationalised, *e.g.* $\frac{1}{2}a\sqrt{6}$ is given and not $a\sqrt{\frac{3}{2}}$; also the vertical angle of an isosceles triangle has been reckoned as twice a half angle].

It is impossible to lay down any hard and fast rule as to the exact degree of reliability of the data in any question. Sometimes they are clearly in "round numbers"; often it is possible to make a shrewd guess at their accuracy, and their very form may guide our impressions [for instance, when all the data are given to two significant figures, one might quite reasonably question a third significant figure in the answer (of course it may be right or it may be wrong, but it is certainly questionable); also "about 3·5" means within half a tenth of a unit (of three and a half) either way, and implies a greater degree of accuracy than "about $3\frac{1}{2}$," which is only something between the extremes $3\frac{1}{4}$ and $3\frac{3}{4}$, though quite probably it is nearer the middle than either end]. Often an answer to many significant figures

is manifestly absurd; we may have used rough values of constants in our computations (and then, of course, pretence to greater accuracy is unwise). Sometimes the instruments, likely to be used to obtain the data, may be subject to so many possible errors that a great number of significant figures is singularly inappropriate; it is easy to imagine cases in which it is a question of life and death, where a quick distinctly approximate answer is (to say the least of it) more valuable than a multitude of figures obtained laboriously, slowly and too late. Further, the ideas of the setter as to the reliability of the data in his question may not be exactly those of its solver (when there can be a doubt, it is the business of the solver to state *clearly* and *briefly* the reasons for the style of his answer; in the following "Answers" such notes have seldom been given, but their omission should not be regarded as a suggestion that they are not in some cases necessary); but, if definite instructions are given in the question as to the degree of accuracy demanded, it is then not a question for the solver's judgment at all. Clearly, where numerical work does not come in, there should be exact agreement or the reason for the discrepancy should be found. [This applies to all calculations of a general type]

In square brackets have been given notes and some figures (which may be useful as checks).

EXAMPLES 40 a. (Pages 430 to 433.)

1. $\frac{1}{2}ab(s+d)$. **2.** Rectangle; $a^2\sqrt{2}$ **3.** $\frac{1}{2}a^2\sqrt{2}$

5. $\sqrt{a^2+b^2+c^2}$, $a\sqrt{3}$ **6.** $a\sqrt{3}$, $a\sqrt{2}$; $\frac{1}{2}a^2\sqrt{6}$ **7.** $\frac{3}{2}a$. **8.** $\frac{1}{2}a^2\sqrt{2}$

9. $\frac{1}{3}a\sqrt{19}$ and $\frac{1}{3}a\sqrt{22}$ **10.** $\frac{1}{2}a\sqrt{b^2+c^2}$; $\frac{1}{2}b\sqrt{a^2+c^2}$.

11. $\sqrt{l^2-\frac{1}{3}a^2}$. **12.** $2\sin^{-1}(\frac{1}{3}\sqrt{3})=$ about $70°\ 32'$

13. $2\sin^{-1}(\frac{1}{6}\sqrt{2})=$ about $27°\ 16'$ **14.** $\frac{1}{2}a\sqrt{2}$, $\frac{1}{2}a^3$, $\frac{3}{4}a^2\sqrt{3}$ **15.** $\frac{1}{4}a\sqrt{6}$

16. $\frac{1}{8}a\sqrt{33}$, $\frac{1}{8}a\sqrt{65}$, $\frac{7}{8}a$. **17.** $\frac{1}{2}\sqrt{-l^2+m^2+n^2-a^2+b^2+c^2}$, etc.

18. $2\sin^{-1}\dfrac{x}{\sqrt{a^2+b^2+c^2}}$, where x is a or b or c.

19. $2\operatorname{cosec}^{-1}\sqrt{3}$, $\sec^{-1}\sqrt{3}$ (twice); or about $70°\ 32'$, about $54°\ 44'$ (twice).

20. $\frac{1}{3}\sqrt{3(l^2+m^2+n^2)-(a^2+b^2+c^2)}$; $\frac{1}{4}$ the same square root.

21. $2\sin^{-1}\left(\dfrac{b\sin\theta}{\sqrt{a^2+b^2}}\right)$; another form is $\cos^{-1}\left(\dfrac{a^2+b^2\cos 2\theta}{a^2+b^2}\right)$. **22.** $\frac{1}{3}a\sqrt{6}$, a

EXAMPLES 40 b. (Pages 433 to 435.)

9. $\frac{1}{6}a^3$.

EXAMPLES 41 a. (Pages 447 to 453.)

2. A right-angle. **3.** About 33° $1\frac{1}{2}'$ [$\tan^{-1} 0{\cdot}65$].
4. About 78° 41′ [$\tan^{-1} 5$]; about 19 ft. 6 in., about 72° 54′ [$\operatorname{cosec}^{-1} \frac{1}{3}\sqrt{104}$].
5. About 39° 48′ **6.** About 70° 32′ [$\sec^{-1} 3$].
7. About 57° 02′ and 32° 58′
8. 25·6″, 9·3″, 22·1″, 23·9″, about 30° 30′, about 54° 20′ **9.** About 24° 09′
10. About 125°(+); $0{\cdot}0589a^3(+)$ [about 125° 16′, $\frac{1}{24}a^3\sqrt{2}$].
11. About 15° 30′ **12.** (1) About 37° 30′; (2) about 61° 08′
13. About 20° 42′ and about 37° 46′
14. (1) About 36° 59′; (2) about 46° 13′ **15.** About 16° 41′
16. 3 ft., 3 ft.; 45° **17.** (1) About 65° $22\frac{1}{2}'$; (2) about 57° 03′
18. (1) About 49° 50′ and about 62° 43′; about 45° 19′ **19.** About 54° 44′
20. 8 cm., 6·9 cm.(+), 4 cm., 6·9 cm.(+).
21. About 7·45 inches, about 73° 34′ **22.** About 19° 59′
23. 69° South of West, about 1 mile (very nearly). **24.** About 32° $57\frac{1}{2}'$
25. About 17° 14′ **26.** About 79° 16′ **27.** 30° **28.** 300 ft.
29. About 11° 53′ **30.** About 35° 16′, about 54° 44′
31. (1) About 28° 37′; (2) about 65° 23′; (3) about 17° 19′
32. Between 4·3″ and 4·4″ [4·36″].
33. About 37° 46′ **34.** 3·62″(−). **35.** 60° **36.** About 126 ft.
37. About $71\frac{1}{2}$°, about 23° [It is ludicrous to pretend to great nicety here.]

EXAMPLES 41 b. (Pages 454 to 456.)

1. (1) $\sqrt{b^2+c^2}$, (2) $\sqrt{c^2+a^2}$, (3) $\sqrt{a^2+b^2}$.
2. $\tan^{-1}\dfrac{c}{a-2d}$. [Naturally b does not come in.]
3. (1) $\cot^{-1}\sqrt{2}=$ about 35° 16′; (2) $\tan^{-1}\sqrt{2}=$ about 54° 44′
4. $\tan^{-1}\sqrt{2}=$ about 54° 44′ **5.** $\frac{1}{2}a\sqrt{2}$ **6.** $\frac{1}{3}(h_1+h_2+h_3)$.
7. $h_1-h_2+h_3$ **8.** $R-\sqrt{R^2-2hr+h^2}$ **9.** $\frac{1}{3}\sqrt{b^2+4c^2}$
10. $\cos^{-1}\dfrac{a^2-b^2}{\sqrt{(a^2+b^2)(a^2+b^2+c^2)}}$ **11.** $\dfrac{ac^2}{b^2+c^2}$ **12.** $\frac{1}{3}a\sqrt{\dfrac{3l^2-a^2}{4l^2-a^2}}$
14. $2\sin^{-1}\left(\dfrac{b\cos\theta}{\sqrt{a^2+b^2}}\right)$; or, in another form, $\cos^{-1}\left(\dfrac{a^2-b^2\cos 2\theta}{a^2+b^2}\right)$.
15. $x\sin\alpha$, $\frac{1}{2}x\sin\theta\cos\alpha$; $\frac{1}{5}\sqrt{5}(2\sin\alpha\sim\sin\theta\cos\alpha)$.
16. $\sqrt{(a+f)^2+(b+g)^2+(c-h)^2}$. **17.** $\sqrt{(a+f)^2+(b+g)^2+(c+h)^2}$.
18. $k-r+\dfrac{r}{f}\sqrt{(a+p)^2+(b+q)^2}$.

EXAMPLES 41 c. (Pages 457 to 460.)

27. 1·99″ (very nearly), $\sqrt{[\frac{95}{24}]}$.

EXAMPLES 42. (Pages 462 to 463.)

1. About $71\frac{1}{2}°$ [*N.B.* $71\frac{1}{2}$ indicates approximation better than 71·5]

2. 400 ft.(–). [Clearly it depends upon the reliability of the data whether 397 ft.(+) can safely be given.]

3. About 2240 ft., about 1 in 3·6 (horizontal). **4.** $18\frac{1}{2}°$; about 1 in 8

5. The following answers (obtained from a small figure) are necessarily approximate : 1 in 4, 1 in $1\frac{3}{4}$, 14°, 30°, 1 in 4·4, 1 in 2

6. 30°, about $26\frac{1}{2}°$ [About 5° 44′ and about 5° 43′.]

7. 1056 ft., 330 ft., $1\frac{1}{4}$ miles, 1 in 4·4

EXAMPLES 43. (Pages 471 to 477.)

11. $a\sqrt{2}$ **12.** Eight of $\frac{1}{2}a$ and four of $\frac{1}{2}a\sqrt{2}$ **13.** $\frac{1}{3}a\sqrt{2}$

14. $\frac{1}{2}a\sqrt{2}$ **15.** 10 sec 15° cm., or between 10·3 cm. and 10·4 cm

17. $\frac{1}{2}a\sqrt{2}$. **18.** 60° **19.** $\frac{1}{4}a\sqrt{6}$ **21.** $\frac{27}{26}V$.

22. $\frac{1}{16}a^2\sqrt{11}$ **23.** 2 cm.

24. (*a*) 8 : 1 ; (*b*) 4 : 1 ; (*d*) $\frac{1}{4}a^2\sqrt{3}$; (*e*) $\frac{1}{3}a\sqrt{6}$; (*f*) same as preceding.

25. (*a*) $\frac{1}{6}a^3$; (*b*) twice ; (*c*) 45° ; (*d*) a and $a\sqrt{2}$; (*e*) about 70° 32′, about 109° 28′ ; (*f*) 120° ; (*g*) 90° ; (*h*) (1) 60°, (2) 90° ; (*i*) about 109° 28′ (or about 70° 32′).

26. (*a*) $a-\frac{1}{3}s\sqrt{6}$; (*b*) $\frac{1}{2}s\sqrt{2}$ and s ; (*c*) (1) $\frac{1}{4}s^2\sqrt{2}$, (2) $\frac{2}{3}a^2\sqrt{2}$; (*d*) 6, 8 ; (*e*) s ; (*f*) $\frac{1}{2}s$; (*g*) $s\sqrt{2}$; (*h*) $\frac{1}{2}s\sqrt{6}$; (*i*) $\frac{4}{3}a\sqrt{3}$, $2a$; (*j*) $\frac{1}{24}s^3\sqrt{2}$; (*k*) $\frac{1}{2}s^3\sqrt{2}$; (*l*) half ; (*m*) $\frac{16}{9}a^3\sqrt{3}$

32. $\frac{1}{2}s\sqrt{6}$, 0·61″ **33.** $\frac{5}{12}a^3(3+\sqrt{5})$ or $\frac{10}{3}a^3\sin^2 54°$ or $\frac{5}{3}a^3(1+\sin 18°)$.

EXAMPLES 44 a. (Pages 484 to 486.)

1. $1\frac{1}{2}$″ **2.** 72″, 81″, 36″

3. Between 1·3 cm. and 1·4 cm. [$4-\sqrt{7}$]. **4.** About $29\frac{1}{2}$″

5. About $85\frac{1}{2}$ cm., 16·2 cm.(–). **6.** 0·04″

7. $66\frac{1}{2}$ ft.(–) [$106-28\sqrt{2}$]. **8.** 2″, 8″, 9″, 15″ **9.** $3\frac{1}{2}$″

10. About 8·49 cm. **11.** About 8·54 cm. **12.** About 9·53 cm.

13. 2·63″(+), 1·25″(–). [The sides of the glass do not slope enough to show that this is less than the radius of the bottom of the glass.]

14. 7 cm. or 15 cm. **15.** 6 cm. **16.** About $4\frac{3}{8}$″

17. (1) 1728, (2) 2120 **18.** $2\frac{3}{4}$″

EXAMPLES 44 b. (Pages 487 to 492.)

2. $R - \sqrt{R^2 - r^2}$.
3. $R + \sqrt{R^2 - r^2}$.
4. $\frac{1}{2}a\sqrt{3}$ and $\frac{1}{2}a$.
5. $4\pi\sqrt{Rr}$.
6. $r - R + \sqrt{R^2 - r^2}$.
8. $\frac{1}{6}a\sqrt{3}$
9. $\frac{1}{5}a\sqrt{5}$
10. $\frac{1}{6}a\sqrt{2}$
11. $\frac{s}{2R}\sqrt{4R^2 - s^2}$.
12. Former; $d(1 + \frac{1}{3}\sqrt{6})$, $d(1 + \frac{1}{2}\sqrt{2})$.
13. 8, 272
14. $\frac{1}{2}(R + r + c)$, $\frac{1}{2}(R - r - c)$.
15. $\frac{1}{2}a\sqrt{2}$, $\frac{1}{6}a\sqrt{6}$
16. $\frac{r}{h}(\sqrt{h^2 + r^2} - r)$.
17. $\frac{1}{2}\{R + r \pm c\}$, $\frac{1}{2}\{c \pm (R - r)\}$.
18. $\frac{1}{2}(c + R + r)$, both internally; $\frac{1}{2}(c - R - r)$, both externally; $\frac{1}{2}\{c + (R - r)\}$, one externally and the other internally.
20. $r\sqrt{2}$
21. $a(\sqrt{3} - 1)$.
23. $\frac{h^2 + k^2}{2h}(\sqrt{2} - 1)$.
24. Below by $0{\cdot}136r$.
25. $2r(1 + \sqrt{6})$.
27. $R(2 - \sqrt{3})$.
28. $\frac{1}{2}(\sqrt{3} - 1)R$.
29. $(3\pi + 2) : \pi$.
30. $2\pi R$.
31. $\frac{1}{4}a\sqrt{6}$
32. $\frac{1}{12}a\sqrt{6}$
33. $\frac{1}{3}R$, $\frac{1}{16}(\sqrt{33} \pm 1)R$.
34. $\frac{r(k - r) + R\sqrt{R^2 - 2rk + k^2}}{R^2 - r^2}$.

EXAMPLES 45 a. (Pages 502 to 512.)

1. 1·4 cm.(+) or 8·9 cm.(+); about 4·7 cm. for maximum capacity, which is about 1825 cu. cm.
2. 117 cu. cm.
3. 11·52 cu. in., 39·2 sq. in.
4. 39 cu. cm.
5. 402 cu. in.
6. (a) About 1400 ft. [1391(−)]; (b) 1360 cu. ft.
7. 2 cu. ft., $3\frac{1}{3}$ cu. ft., $\frac{2}{3}$ cu. ft.
8. 13 oz.
9. (b) 4·8″, (c) 4·1″(−).
10. 234 cu. in., 293·4 sq. in.; same volume, 333·9 sq. in.
11. (c) 4·4 mm.
12. 4·3 cm.(+); minimum surface if "Height = diameter.
13. 54 Kgr.
14. (1) 1500 cu. cm., (2) about 3160 sq. cm.
15. 22·1″ (last part).
16. About 141·37 cu. in. [45π].
17. 3510 cu. in.
18. About 3900 cu. ft., 27 tons.
19. $16\frac{2}{3}$ cu. yds., about $8\frac{1}{3}$ hundred tons.
20. 5·4 tons(−).
21. 168 cu. in., about $254\frac{1}{2}$ sq. in.
23. 13·9 cm., 400 cu. cm., 360 sq. cm.
24. 10·9 cm. (very nearly); about 363·6 cu. cm.
25. 100 cu. in.(+) [100·352]; between 143 sq. in. and 144 sq. in. [143·36]; 10·4 in.(−).
26. 299 sq. ft.
27. 1800 cu. cm., about 1254 sq. cm.
28. Volume between 366 cu. in. and 367 cu. in., or about 10·6 pints; surface area about 324 sq. in.

29. (*c*) 72 cu. cm. **30.** $194\frac{2}{3}$ cu. in. **31.** 6 lb. 9 oz.
32. About $1\frac{3}{4}$ pints. **33.** Volume of bath = 18 gallons(–).
34. Roughly 374 sq. in., 3 gallons. **35.** ·9 **36.** 717 gallons (very nearly).
37. About 253·7 sq. in., about 141·37 cu. in.
38. 8″, 448π cu. in., 84 sq. in. **39.** About 1·54 cu. in.

EXAMPLES 45 b. (Pages 512 to 517.)

2. $\frac{1}{4}a^2l\sqrt{3}$ **3.** $\frac{1}{27}$ **4.** (1) $6ah+3a^2\sqrt{3}$, (2) $\frac{3}{2}a^2h\sqrt{3}$
6. $\frac{1}{2}r^2l(9-\pi\sqrt{3})$. **7.** 1 : 6
8. (1) $\{2l(6+3\sqrt{3}+\pi)+r(2\pi+3\sqrt{3})\}:\{2l(2\pi+3\sqrt{3})+r(4\pi-3\sqrt{3})\}$;
(2) $\{4\pi-3\sqrt{3}\}:\{2\pi+3\sqrt{3}\}$.
10. $\frac{1}{3}\pi h^2(a+2b)$. **11.** Same as preceding answer.
12. (1) $a^2\sqrt{3}$; (2) $\frac{1}{12}a^3\sqrt{2}$ **13.** $2a^2\sqrt{3}$, $\frac{1}{3}a^3\sqrt{2}$
16. $\frac{1}{6}a^3$, $\frac{1}{3}a^3$. **17.** 7040 cu. ft. **19.** $\frac{7}{72}\pi h^3$.
20. $2\sqrt{2}-2+\sqrt{3}-\frac{2}{3}\sqrt{6}$ times area of original, and $\frac{7}{3}(\sqrt{2}-1)$ times volume of original; $s=a(\sqrt{2}-1)$.
21. $\frac{1}{48}a^3$, $\frac{5}{6}a^3$, $(3+\sqrt{3})a^2$ **22.** 3 : 1; $\frac{2}{3}$; $\frac{1}{9}(6+\sqrt{3})$; $\frac{8}{9}$
23. $(\sqrt[3]{2}-1):1$ **24.** $2\pi b^3$. [Notice that the value of *a* does not matter.]
25. $6\pi b^2\sqrt{2}$ [Again the value of *a* is immaterial.]
26. For maximum $h=\frac{1}{3}H$. **27.** 4π cosec 15° sq. cm.
28. $\{\sqrt{r+l}\}:\{\sqrt{2l}-\sqrt{r+l}\}$. **29.** $\frac{1}{3}\pi k^3(3+2\sqrt{3})$.
30. (1) 42π sq. units, 36π cu. units; (2) 96π, 72π, 24π, 12π;
(3) 96π, 72π, $\frac{81}{2}\pi$, 21π.
31. $\dfrac{\pi R^3(R-r+c)^2}{3(c-R+r)(R-r)}$. **33.** $\dfrac{4\pi\Delta^2}{3a}$ and two similar expressions.

EXAMPLES 46 a. (Pages 523 to 526.)

1. About 0·44 oz. **2.** About $1{\cdot}6\times10^{10}$ sq. mi. **3.** 48%.
4. About $15\frac{1}{3}$ min. **5.** About 1·18″ **6.** 2·6 mm.(–), 970 cu. cm. mark.
7. About $53\frac{1}{3}$ cu. in; about $38\frac{1}{2}$ sq. in.; diameter about $4\frac{2}{3}$″ (very near).
8. (*d*) 2·2 cm.(–). **9.** More. [Slightly, as a matter of fact by about 1%.]
10. 5″ **11.** X by between 205 gr. and 206 gr., Y by the same [$\frac{128}{3}\pi$].
12. About $18\frac{1}{2}$ ft., between 148 lb. and 149 lb.
13. Between 5·8″ and 5·9″ [$8(\sqrt{3}-1)$].
14. Less than one two-thousand-millionth part of the Sun's radiation reaches the Earth $\left[\dfrac{1}{2{,}000{,}000{,}000}\right]$.
15. 1910 lb.(–). **16.** Roughly 6 million square miles. **17.** 8 sq. ft.
18. Between 0·78″ and 0·79″; about 351 cu. in., about 343 cu. in. [Doubtful data really.]
19. (*a*) About $2785\frac{1}{2}$ cu. cm.; (*b*) 66·5%; (*c*) about 3·4 Kgr.,
(*d*) 410 sq. cm.(–). **20.** $4\frac{1}{2}$ cm.

EXAMPLES 46 b. (Pages 527 to 533.)

1. $\frac{1}{6}r$. **2.** About 288 cm. **5.** 0·4

6. $\frac{1}{y-2}$ of the diameter of the sphere. **7.** $a\sqrt{\frac{6}{\pi}}$, more.

8. 124% ; 81 : 100 **9.** $\sqrt{2}-1$, 48% and 52%.

10. About 56° and 66° **12.** Rhombus larger by 0·5%.

13. $42\frac{3}{4}$% [very nearly].

14. (1) $2\pi r^2(5+4\sqrt{2})$; (2) $\frac{2}{3}\pi r^2(13+4\sqrt{2}+8\sqrt{3}+2\sqrt{6})$;
(3) $8\pi r^2\sqrt[3]{3}$; 117 and 54

15. $\{4R^3-3h^2R+h^3\} : \{3Rh^2-h^3\}$ **16.** $\{4R^2-h^2\} : \{4Rh-h^2\}$

17. $\frac{1}{6}\pi h(3r^2+h^2)$. **18.** (1) 7 : 11 ; (2) 5 : 11 **19.** $\frac{5}{16}$

20. $2\pi(r^2+k^2)$, $\frac{1}{6}\pi k(3r^2+k^2)$. **21.** $\pi(h^2+2r^2)$.

22. $\frac{1}{18}(7\sqrt{3}-12)\pi r^3$. **23.** $\frac{1}{6}(8-5\sqrt{2})\pi r^3$.

24. $\frac{1}{3}\pi\{r^2h-2R^3+(2R^2+r^2)\sqrt{R^2-r^2}\}$. **26.** $h-R+k+\sqrt{R^2-r^2}$.

27. $\frac{1}{3}\pi\{2(R^3+r^3)+2(R^2C+r^2c)+k^2(C+c)\}$
or, in another form, $\frac{1}{3}\pi\{(R+C)^2(2R-C)+(r+c)^2(2r-c)\}$.

28. $\frac{1}{6}\pi\{6k^2p+(H^3-h^3)+3k^2(H-h)\}$.

29. $2\pi r^2(3-\sqrt{2})$, $\frac{1}{3}\pi r^3(7-4\sqrt{2})$. **30.** $\frac{1}{3}\pi r^3(3+4\sqrt{2})$.

EXAMPLES 47. (Pages 535 to 539.)

5. 1000, 10 : 1 **7.** 64%. **8.** Same. **9.** 6″

10. 12 cm., 25 : 18 **11.** 5 cm., 91 : 125 **12.** 6 lb. 14 oz. **13.** 9%.

14. 1 : 24 ; 1 : 576 ; 1 : 13,824 ; 660 sq. ft. **15.** 63%. **16.** 1·71 oz.

17. About $5\frac{1}{2}$% ; about $3\frac{3}{8}$% ; less. **18.** 4%.

19. 64 : 81 : 100 ; 51 : 73 : 100.

20. Roughly $79\frac{1}{2}$: 91 : 100 ; roughly 63 : $82\frac{1}{2}$: 100

21. 63, 79 and 100 **22.** 86%, 93%.

23. 13 ft. $1\frac{1}{2}$ in., 1 ft. $5\frac{1}{8}$ in. ; 48 : 1 ; 15 sq. ft. ; about $\frac{1}{4}$ ton.

24. About 1·26, 1·59(−), 1·91(+). **26.** 40 : 63 : 100 **27.** $9\frac{1}{8}$″

28. 8·8 sq. in.(+). **29.** 0·662″(+).

EXAMPLES 48. (Pages 545 to 550.)

1. $bc=ab=6{\cdot}3$ ft.(+) ; $ca=7{\cdot}2$ ft.(+).

2. Between 8·7 cm. and 8·8 cm., (1) about 20°, (2) about $29\frac{1}{2}$°, (3) about 53°

6. (1) 45° and 47° (very nearly) ; (2) about $25\frac{1}{2}$ ft., 36°(+).

8. About $8\frac{3}{4}$ ft. ; (1) about $19\frac{3}{4}$ ft., (2) $26\frac{1}{2}$°(−). **9.** 32 cu. ft.(+).

11. Between 2·5 ft. and 2·6 ft., about 2·6 ft., about $29\frac{1}{2}$ sq. ft., 16 cu. ft.(−).

12. 104 lb. **13.** 3·20 ft.(+), 4·27 ft.(+), 4·03 ft.(+). [$\sqrt{10{\cdot}25}$... etc.]

15. $7\frac{1}{3}$ cu. ft. **16.** 1·7a, 2·2a, 2·3a^3 ; $\frac{1}{3}(a\sqrt{3}-y)^2$.

19. (The following answers must be rough from such a small figure): About $12\frac{1}{2}$ cm., about 60 sq. cm.

20. About 52° [51° 48′ is the accurate answer].

21. 619 sq. in.(+), 14° (very nearly).

22. $x+6z=6$

EXAMPLES 49. (Pages 554 to 555.)

15. About 0·008 (very near).

EXAMPLES 51. (Pages 570 to 573.)

1. About 10.
2. 720 mi., 570 mi.
3. About 1600 yards. Between 2500 yards and 2600 yards.
4. Yes (but not by much).
5. 20,000 sq. mi.
6. Less than 8·2 million sq. mi.
7. More than 5 million sq. mi.
8. 15 nautical miles [821 − 806].
9. Roughly (1) 1546 mi., (2) 1515 mi., (3) 1524 mi.
10. 211 nautical miles, 46° 02′ S. lat.
11. (1) 1681 nautical miles, (2) 1638 n.m., (3) about 27° 34′, (4) 1654 n.m., (5) 27 n.m., (6) about 53° 41′ N. lat.
12. About 78° 33′ N. lat.

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